ON POINTED HOPF ALGEBRAS ASSOCIATED WITH THE MATHIEU SIMPLE GROUPS

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Abstract. Let G be a Mathieu simple group, $s \in G$, $\mathcal{O}_s$ the conjugacy class of s and ρ an irreducible representation of the centralizer of s. We prove that either the Nichols algebra $\mathfrak{B}(\mathcal{O}_s, \rho)$ is infinite-dimensional or the braiding of the Yetter-Drinfeld module $M(\mathcal{O}_s, \rho)$ is negative. We also show that if $G = M_{22}$ or M_{24}, then the group algebra of G is the only (up to isomorphisms) finite-dimensional complex pointed Hopf algebra with group-likes isomorphic to G.

Introduction

This article contributes to the classification of finite-dimensional complex pointed Hopf algebras H whose group of group-like elements $G(H)$ is isomorphic to a Mathieu simple group: M_{11}, M_{12}, M_{22}, M_{23} or M_{24}.

The crucial step, in order to classify finite-dimensional complex pointed Hopf algebras with a fixed $G(H) = G$, is to determine when a Nichols algebra of a Yetter-Drinfeld module over G is finite-dimensional – see [AS2].

The irreducible Yetter-Drinfeld modules over G are determined by a conjugacy class $\mathcal{O}$ of G and an irreducible representation ρ of the centralizer G^s of a fixed $s \in \mathcal{O}$. Let $M(\mathcal{O}, \rho)$ be the corresponding Yetter-Drinfeld module and let $\mathfrak{B}(\mathcal{O}, \rho)$ denote its Nichols algebra.

The classification of finite-dimension Nichols algebras over an abelian group G follows from [AS1, H1, H2]; this leads to substantial classification results of pointed Hopf algebras H with abelian $G(H)$ – see [AS3]. The next problem is to discard irreducible Yetter-Drinfeld modules over a finite non-abelian group containing a braided vector subspace with infinite-dimensional Nichols algebra. It is natural to begin by simple or almost simple groups; see [AZ, AF1, AF2, AFZ], for $\mathbb{A}_n$ or $\mathbb{S}_n$; [FGV] for $\mathbf{GL}(2, \mathbb{F}_q)$ or $\mathbf{SL}(2, \mathbb{F}_q)$; and [FV] for $\mathbf{PGL}(2, \mathbb{F}_q)$ or $\mathbf{PSL}(2, \mathbb{F}_q)$. We plan to consider the other sporadic groups in [AFGV].

Let us say that $M(\mathcal{O}, \rho)$ has *negative braiding* if the Nichols algebra of any braided subspace corresponding to an abelian subrack is (twist-equivalent to) an exterior algebra.

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G	j	$\vert s_j\vert$	**Centralizer**	**Representation**	$\dim M(\mathcal{O}_{s_j},\rho)$
M_{11}	4	4	$\langle x\rangle \simeq \mathbb{Z}_8$, $x^6 = s_4$	$\nu_2(x) := i$	990
				$\nu_6(x) := -i$	
	6	8	$\langle s_6\rangle \simeq \mathbb{Z}_8$	$\chi_{(-1)}$	990
	7	8	$\langle s_7\rangle \simeq \mathbb{Z}_8$	$\chi_{(-1)}$	990
M_{12}	13	10	$\langle s_{13}\rangle \simeq \mathbb{Z}_{10}$	$\chi_{(-1)}$	9504
M_{23}	12	14	$\langle s_{12}\rangle \simeq \mathbb{Z}_{14}$	$\chi_{(-1)}$	728640
	13	14	$\langle s_{13}\rangle \simeq \mathbb{Z}_{14}$	$\chi_{(-1)}$	728640

TABLE 1. Cases of negative braiding.

We summarize the investigation in this paper in the next statement.

Theorem 1. *Let G be a Mathieu simple group, $s \in G$, $\mathcal{O}_s$ the conjugacy class of s and $\rho \in \widehat{G^s}$. If $\dim \mathfrak{B}(\mathcal{O}_s,\rho) < \infty$, then $(\mathcal{O}_s,\rho)$ is one of the pairs listed in Table 1. In particular, any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq M_{22}$ or M_{24} is necessarily isomorphic to the group algebras $\mathbb{C}M_{22}$ or $\mathbb{C}M_{24}$, respectively.*

The proof of the theorem, as well as the unexplained notation, is contained in sections 2 and 3. In Section 1, we set some notations and collect preliminary results needed in the sequel.

We use GAP [S] to compute the character tables and other computations, such as representatives of conjugacy classes, intersections between centralizers and conjugacy classes, etc. These computations are available at `http://www.mate.uncor.edu/~fantino/GAP/mathieu.htm`. In main body of the paper the phrase "we compute" means that we have performed the computations with the computational algebra system mentioned above.

0.1. **Notations.** We will follow the conventions in [AZ, AF1]. We denote by $\widehat{G}$ the set of isomorphism classes of irreducible representations of a finite group G. We will use the rack notation $x \triangleright y := xyx^{-1}$. We denote by $\mathbb{G}_n$ the group of n-th roots of 1 in $\mathbb{C}$ and $\omega_n := e^{\frac{2\pi i}{n}}$, where $i = \sqrt{-1}$. The representation of the cyclic group $\mathbb{Z}_{2n} = \langle [1]\rangle$ corresponding to $\rho([1]) = \omega_{2n}^n = -1$ will be denoted by $\chi_{(-1)}$.

For $s \in G$ we denote by $\mathcal{O}_s$ (resp. G^s) the conjugacy class (resp. the centralizer) of s in G. For $y \in G^s$, we denote the conjugacy class of y in the group G^s by $\mathcal{O}_y^{G^s}$. Also, for k, with $1 \leq k \leq |\widehat{G^s}|$, we denote the k-th conjugacy class of G^s by $\mathcal{O}_k^{G^s}$.

In the character tables, that we give in Section 2, we include the following information: the first row enumerates the conjugacy classes of the group with the parameter j, the second row gives the order $|s_j|$ of a representative of each conjugacy class, the third row give the order of the centralizer G^{s_j} of s_j in the corresponding Mathieu simple group. Notice that if all the numbers in

the column corresponding to s_j are real, then s_j is *real*, i. e. $s_j^{-1} \in \mathcal{O}_{s_j}$. For simplicity we will omit the cardinal of the conjugacy classes and the order of the centralizers in some character tables. Also, for a complex number z we denote the complex conjugate of z by z' (and not $\bar{z}$) for a better reading of the tables.

1. Preliminaries

1.1. **Yetter-Drinfeld modules over a finite group.** We recall that the irreducible Yetter-Drinfeld module $M(\mathcal{O}, \rho)$, with $\mathcal{O}$ a conjugacy class of G and $\rho = (\rho, V)$ in $\widehat{G^s}$, for a fixed element $s \in \mathcal{O}$, is described as follows. Let $\sigma_1, \dots, \sigma_N$ be a numeration of $\mathcal{O}$ and let $g_j \in G$ such that $g_j \triangleright s = \sigma_j$ for all $1 \le j \le N$. Then $M(\mathcal{O}, \rho) = \oplus_{1 \le j \le N}\, g_j \otimes V$. We will write $g_j v := g_j \otimes v \in M(\mathcal{O}, \rho)$, $1 \le j \le N$, $v \in V$. If $v \in V$ and $1 \le j \le N$, then the action of $g \in G$ is given by $g \cdot (g_j v) = g_l(\gamma \cdot v)$, where $g g_j = g_l \gamma$, for some $1 \le l \le N$ and $\gamma \in G^s$, and the coaction is given by $\delta(g_j v) = \sigma_j \otimes g_j v$. The Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ is a braided vector space with braiding

$$c(g_j v \otimes g_k w) = \sigma_j \cdot (g_k w) \otimes g_j v = g_l(\gamma \cdot w) \otimes g_j v, \tag{1.1}$$

for any $1 \le j, k \le N$, $v, w \in V$, where $\sigma_j g_k = g_l \gamma$ for unique l, $1 \le l \le N$ and $\gamma \in G^s$. Since $s \in Z(G^s)$, the center of the group G^s, the Schur Lemma implies that

$$s \text{ acts by a scalar } q_{ss} \text{ on } V. \tag{1.2}$$

A braided vector space (W, c) is of *diagonal type* if there exists a basis $w_1, \dots, w_\theta$ of W and non-zero scalars q_{ij}, $1 \le i, j \le \theta$, such that $c(w_i \otimes w_j) = q_{ij} w_j \otimes w_i$, for all $1 \le i, j \le \theta$. The *generalized Dynkin diagram* associated with (W, c) of diagonal type as above is the diagram with vertices $\{1, \dots, \theta\}$, where the vertex i is labelled by q_{ii}, and if $q_{ij} q_{ji} \ne 1$, then the vertices i and j are joined by an edge labelled by $q_{ij} q_{ji}$, i. e.

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see [H2]. A braided vector space (W, c) of diagonal type is of *Cartan type* if q_{ij} is a root of 1 for all i, j, $1 \le i, j \le \theta$, $q_{ii} \ne 1$ for all i, $1 \le i \le \theta$, and there exists $a_{ij} \in \mathbb{Z}$, with $-\operatorname{ord} q_{ii} < a_{ij} \le 0$, such that $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for all $1 \le i \ne j \le \theta$ – see [AS1]. Set $a_{ii} = 2$ for all $1 \le i \le \theta$. Then $(a_{ij})_{1 \le i, j \le \theta}$ is a generalized Cartan matrix.

1.2. **Tools.** We state the principal tools that we will use in Section 2.

Lemma 1.1. [AZ, Remark 1.1]. *Let (W, c) be a braided vector space, U a subspace of W such that $c(U \otimes U) = U \otimes U$. If $\dim \mathfrak{B}(U) = \infty$, then $\dim \mathfrak{B}(W) = \infty$.* □

This result implies that if $\mathcal{O}_{\mathrm{id}}$ is the conjugacy class of the identity element of G and $\rho \in \widehat{G}$, then $\dim \mathfrak{B}(\mathcal{O}_{\mathrm{id}}, \rho) = \infty$. Thus, we omit to consider this conjugacy class in the proofs of the Theorems 2.1, 2.3, 2.5, 2.7 and 2.9.

Theorem 1.2. [H1, Th. 4]*, see also* [AS1, Th. 1.1]*. Let* (W, c) *be a braided vector space of Cartan type. Then* $\dim \mathfrak{B}(W) < \infty$ *if and only if the Cartan matrix is of finite type.* □

We say that $s \in G$ is *real* if it is conjugate to s^{-1}; if s is real, then the conjugacy class of s is also said to be *real*.

Lemma 1.3. *If s is real and* $\dim \mathfrak{B}(\mathcal{O}_s, \rho) < \infty$*, then* $q_{ss} = -1$. □

If $s^{-1} \neq s$, this is [AZ, Lemma 2.2]; if $s^2 = \mathrm{id}$, then $q_{ss} = \pm 1$, but $q_{ss} = 1$ is excluded by Lemma 1.1. Notice that $q_{ss} = -1$ implies that s has even order. The following is a generalization of Lemma 1.3. See [AF2, Lemmata 1.8 and 1.9] or [FGV, Corollary 2.2].

Lemma 1.4. *Let G be a finite group, $s \in G$, $\mathcal{O}$ the conjugacy class of s and* $\rho = (\rho, V) \in \widehat{G^s}$ *such that* $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$*. Assume that there exists an integer k such that $s^k \in \mathcal{O}$ and $s^k \neq s$.*

(a) *If* $\deg \rho > 1$*, then* $q_{ss} = -1$*.*
(b) *If* $\deg \rho = 1$*, then either* $q_{ss} = -1$ *or* $q_{ss} \in \mathbb{G}_3 - 1$*.*

On the other hand, if $s^{k^2} \neq s$*, then* $q_{ss} = -1$*.* □

The next important tool follows from [H2].

Lemma 1.5. *Let W be a Yetter-Drinfeld module, $U \subseteq W$ a braided vector subspace of diagonal type of W such that q_{ii} is a root of 1 for all i, and let $\mathcal{G}$ be the generalized Dynkin diagram corresponding to U. If $\mathcal{G}$ contains an r-cycle with $r > 3$, or a vertex with valency greater than 3, then the Nichols algebra* $\mathfrak{B}(U)$ *is infinite-dimensional. Hence,* $\dim \mathfrak{B}(W) = \infty$*.* □

Abelian subspaces of a braided vector space. As in [AF1, AF2], in a first step we look for braided subspaces W of diagonal type of $M(\mathcal{O}, \rho)$ whose Nichols algebra is infinite-dimensional.

Let $(X, \triangleright)$ be a rack, see for example [AG]. Let $q : X \times X \to \mathbb{C}^{\times}$ be a rack 2-cocycle and let $(\mathbb{C}X, c_q)$ be the associated braided vector space, that is $\mathbb{C}X$ is a vector space with a basis e_k, $k \in X$, and $c_q(e_k \otimes e_l) = q_{k,l} e_{k \triangleright l} \otimes e_k$, for all k, $l \in X$. Let us say that a subrack T of X is *abelian* if $k \triangleright l = l$ for all k, $l \in T$. If T is an abelian subrack of X then $\mathbb{C}T$ is a braided vector subspace of $(\mathbb{C}X, c_q)$ of diagonal type.

We shall say that $(\mathbb{C}X, c_q)$ is *negative* if for any abelian subrack T of X $q_{kk} = -1$ and $q_{kl} q_{lk} = 1$ for all $k, l \in T$ (hence $\mathfrak{B}(\mathbb{C}T)$ is twist-equivalent to an exterior algebra and $\dim \mathfrak{B}(\mathbb{C}T) = 2^{\operatorname{card} T}$).

Let G be a finite group, $\mathcal{O}$ a conjugacy class in G, $\rho \in \widehat{G^s}$, with $s \in \mathcal{O}$ fixed. As in subsection 1.1, we fix a numeration $\sigma_1 = s, \dots, \sigma_N$ of $\mathcal{O}$ and $g_k \in G$ such that $g_k \triangleright s = \sigma_k$ for all $1 \le k \le N$. Let $I \subset \{1, \dots, N\}$ and $T := \{\sigma_k : k \in I\}$. We characterize when T is an abelian subrack of $\mathcal{O}$. Let

$$\gamma_{k,l} := g_l^{-1} \sigma_k g_l, \qquad k, l \in I. \tag{1.3}$$

Then the following are equivalent:

(a) $\sigma_k \triangleright \sigma_l = \sigma_l$ (i. e. σ_k and σ_l commute) and (b) $\gamma_{k,l} \in G^s$.

Assume that (a) (or, equivalently, (b)) holds for all $k, l \in I$; then $\gamma_{k,l} \in \mathcal{O}_s \cap G^s$. Let V be the vector space affording ρ. Let $v_1, \dots, v_R$ be simultaneous eigenvectors of $\gamma_{k,l}$, $k, l \in I$. Because of (1.1), we have that

$$W = \mathbb{C} - \text{span of } g_k v_j, \quad k \in I, 1 \le j \le R,$$

is a braided subspace of diagonal type of dimension $R \operatorname{card} T$. Notice that R depends not only on T but also on the representation ρ; for instance if ρ is a character then $R = 1 = \dim V$, and $M(\mathcal{O}, \rho)$ is of rack type.

Lemma 1.6. *Assume that σ_k, $\sigma_l \in \mathcal{O}$ commute and $\deg \rho = 1$. Then the scalar $\rho(\gamma_{k,l})$ does not depend on g_k and g_l.*

Proof. Let $\widetilde{g_k}, \widetilde{g_l} \in G$ such that $\widetilde{g_k} \triangleright s = \sigma_k$ and $\widetilde{g_l} \triangleright s = \sigma_l$. Thus, $\widetilde{g_k} = g_k \eta_k$ and $\widetilde{g_l} = g_l \eta_l$, with $\eta_k, \eta_l \in G^s$. If we call $\widetilde{\gamma_{k,l}} := \widetilde{g_l}^{-1} \widetilde{g_k}\, s\, \widetilde{g_k}^{-1} \widetilde{g_l}$, then

$$\widetilde{\gamma_{k,l}} = \eta_l^{-1}\, g_l^{-1} \sigma_k g_l\, \eta_l = \eta_l^{-1}\, \gamma_{k,l}\, \eta_l.$$

Hence, $\rho(\widetilde{\gamma_{k,l}}) = \rho(\eta_l)^{-1} \rho(\gamma_{k,l}) \rho(\eta_l) = \rho(\gamma_{k,l})$, since $\deg \rho = 1$. □

In view of this result, we can choose $g_1 = \mathrm{id}$, the identity of the group G.

Remark 1.7. If $\deg \rho = 1$, then the condition of negative braiding is equivalent to (i) $\rho(\gamma_{k,k}) = -1$ and

(ii) for every commuting pair σ_k, $\sigma_l \in \mathcal{O}$, it holds $\rho(\gamma_{k,l}\gamma_{l,k}) = 1$.

The next result is useful in order to prove that $M(\mathcal{O}, \rho)$ has negative braiding when ρ is an one-dimensional representation.

Lemma 1.8. *The condition* (ii) *given above is equivalent to*

(ii)' *for every $\sigma_t \in \mathcal{O} \cap G^s$, it holds $\rho(\gamma_{1,t}\gamma_{t,1}) = 1$.*

Proof. Obviously, (ii) implies (ii)'. Reciprocally, assume that (ii)' holds. Let σ_k, $\sigma_l \in \mathcal{O}$ that commute. Then, $\gamma_{k,l} = g_l^{-1} g_k\, s\, g_k^{-1} g_l$, $\gamma_{l,k} = g_k^{-1} g_l\, s\, g_l^{-1} g_k$ are in $\mathcal{O} \cap G^s$. Hence, $\gamma_{k,l} = \sigma_t$, for some $1 \le t \le N$. By Lemma 1.6, $\rho(\gamma_{k,l}\gamma_{l,k}) = \rho(\gamma_{1,t}\gamma_{t,1})$, and the result follows. □

1.3. **Criterions from non-abelian subracks.** We will mention here some criterions that allow to decide the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{O},\rho)$ using non-abelian subracks of $\mathcal{O}$. These criterions were developed in [AF3] using important results of [AHS].

Let $p>1$ be an integer. A family $(\sigma_i)_{i\in\mathbb{Z}_p}$ of distinct elements of a group G is of *type* $\mathcal{D}_p$ if $\sigma_i\triangleright\sigma_j=\sigma_{2i-j}$, $i,j\in\mathbb{Z}_p$. Let $(\sigma_i)_{i\in\mathbb{Z}_p}$ and $(\tau_i)_{i\in\mathbb{Z}_p}$ be two families of type $\mathcal{D}_p$ in G, such that $\sigma_i\neq\tau_j$ for all $i,j\in\mathbb{Z}_p$, we say that $(\sigma,\tau):=(\sigma_i)_{i\in\mathbb{Z}_p}\cup(\tau_i)_{i\in\mathbb{Z}_p}$ is of *type* $\mathcal{D}_p^{(2)}$ if

$$\sigma_i\triangleright\tau_j=\tau_{2i-j},\quad \tau_i\triangleright\sigma_j=\sigma_{2i-j},\quad i,j\in\mathbb{Z}_p. \tag{1.4}$$

Lemma 1.9. [AF3, Cor. 2.9] *Let p be an odd prime, $(\sigma_i)_{i\in\mathbb{Z}_p}$ a family of type $\mathcal{D}_p$ in a finite group G and $\rho\in\widehat{G^{\sigma_0}}$. If there exists k, such that $\sigma_0^k\in\mathcal{O}$, the conjugacy class of σ_0, and $q_{\sigma_0\sigma_0}=-1$, then* $\dim\mathfrak{B}(\mathcal{O},\rho)=\infty$. □

Let $\mathcal{O}_4^4$ be the conjugacy class of the 4-cycles in the symmetric group $\mathbb{S}_4$. We will say that a family $(\sigma_i)_{1\leq i\leq 6}$ of distinct elements of a group G is of *type* $\mathfrak{O}$ if $(\sigma_i)_{1\leq i\leq 6}$ form a rack isomorphic to $\mathcal{O}_4^4$. We call such a rack an *octahedral* rack. Let σ_i, $\tau_i\in G$, $1\leq i\leq 6$, all distinct; we say that $(\sigma,\tau):=(\sigma_i)_i\cup(\tau_i)_i$ is of *type* $\mathfrak{O}^{(2)}$ if $(\sigma_i)_i$ and $(\tau_i)_i$ are both of type $\mathfrak{O}$, and

$$\sigma_i\triangleright\tau_j=\tau_{i\triangleright j},\quad \tau_i\triangleright\sigma_j=\sigma_{i\triangleright j},\quad 1\leq i,j\leq 6, \tag{1.5}$$

where $\triangleright$ in the subindex denotes the operation of rack in the octahedral rack.

We state the main tool from this non-abelian rack – see [AF3, Th. 4.11].

Lemma 1.10. *Let G be a finite group, $(\sigma_i)_i\cup(\tau_i)_i$ a family of type $\mathfrak{O}^{(2)}$ in $\mathcal{O}$ the conjugacy class of σ_1, and $g\in G$ such that $g\triangleright\sigma_1=\tau_1$. Let $\rho=(\rho,V)\in\widehat{G^{\sigma_1}}$ with $q_{\sigma_1\sigma_1}=-1$. If there exist v, $w\in V-0$ such that*

$$\rho(\sigma_6)v=-v,\qquad \rho(g^{-1}\sigma_1 g)w=-w,$$
$$\rho(\tau_1)v=-v,\qquad \rho(g^{-1}\sigma_6 g)w=-w,$$

then $\dim\mathfrak{B}(\mathcal{O},\rho)=\infty$. □

The following is a useful consequence of this result.

Lemma 1.11. [AF3, Cor. 4.12] *Let G be a finite group, $(\sigma_i)_i\cup(\tau_i)_i$ a family of type $\mathfrak{O}^{(2)}$ in $\mathcal{O}$ the conjugacy class of σ_1, and $\rho\in\widehat{G^{\sigma_1}}$, with $q_{\sigma_1\sigma_1}=-1$. If $\sigma_6=\sigma_1^d$ and $\tau_1=\sigma_1^e$, then* $\dim\mathfrak{B}(\mathcal{O},\rho)=\infty$. □

2. Using techniques based on abelian subracks

In this section, we will determine the irreducible Yetter-Drinfeld modules $M(\mathcal{O},\rho)$ with $\dim\mathfrak{B}(\mathcal{O},\rho)=\infty$ by mean of abelian subracks of $\mathcal{O}$. We will consider each simple Mathieu group separately.

j	1	2	3	4	5	6	7	8	9	10
$\lvert s_j\rvert$	1	11	11	4	2	8	8	3	5	6
$\lvert G^{s_j}\rvert$	7920	11	11	8	48	8	8	18	5	6
$\lvert \mathcal{O}_{s_j}\rvert$	1	720	720	990	165	990	990	440	1584	1320
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	-1	-1	2	2	0	0	1	0	-1
χ_3	10	-1	-1	0	-2	B	B'	1	0	1
χ_4	10	-1	-1	0	-2	B'	B	1	0	1
χ_5	11	0	0	-1	3	-1	-1	2	1	0
χ_6	16	A	A'	0	0	0	0	-2	1	0
χ_7	16	A'	A	0	0	0	0	-2	1	0
χ_8	44	0	0	0	4	0	0	-1	-1	1
χ_9	45	1	1	1	-3	-1	-1	0	0	0
χ_{10}	55	0	0	-1	-1	1	1	1	0	-1

Table 2. Character table of M_{11}.

2.1. **The group M_{11}.** The Mathieu simple group M_{11} can be given as a subgroup of $\mathbb{S}_{11}$ in the following form

$$M_{11} := \langle\, (1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)(4,10,5,6)\,\rangle.$$

In Table 2, we show the character table of M_{11}, where $A = (-1 - i\sqrt{11})/2$, $B = i\sqrt{2}$. We take the following elements to be the representatives of the conjugacy classes of M_{11}:

$s_1 :=$ id, $\qquad s_2 := (1,9,7,10,8,11,5,4,3,6,2),$

$s_3 := (1,7,8,5,3,2,9,10,11,4,6),$ $\qquad s_4 := (1,8,2,7)(4,6,10,5),$

$s_5 := (1,2)(4,10)(5,6)(7,8),$ $\qquad s_6 := (1,3,11,6,7,10,4,5)(8,9),$

$s_7 := (1,10,11,5,7,3,4,6)(8,9),$ $\qquad s_8 := (1,6,4)(2,9,7)(8,11,10),$

$s_9 := (1,2,3,4,8)(5,10,7,11,6),$ $\qquad s_{10} := (1,5,8,4,6,9)(2,10,3)(7,11).$

In the following statement, we summarize our study by mean of abelian subracks in the group M_{11}.

Theorem 2.1. *Let $\rho \in \widehat{M_{11}^{s_j}}$, with $1 \le j \le 10$. The braiding is negative in the cases $j = 4$, with $\rho = \nu_2$ or ν_6, $j = 6$, 7 and 10, with $\rho = \chi_{(-1)}$. Otherwise,* $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$.

Proof. From Table 2, we see that for $j = 4$, 5, 8, 9 and 10, s_j is real.

CASE: $j = 8$, 9. By Lemma 1.3, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{11}^{s_j}}$.

CASE: $j = 2$, 3. We compute that s_j^3 and s_j^9 are in $\mathcal{O}_{s_j}$, and $s_j^3 \neq s_j^9$. By Lemma 1.4, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{11}^{s_j}}$, since $|s_j| = 11$.

CASE: $j = 4$. The element s_4 is real and we compute that

$$M_{11}^{s_4} = \langle x := (1,4,7,5,2,10,8,6)(3,11)\rangle \simeq \mathbb{Z}_8,$$

with $x^6 = s_4$. We set $\widehat{M_{11}^{s_4}} = \{\nu_0, \dots, \nu_7\}$, where $\nu_l(x) := \omega_8^l$, $0 \le l \le 7$. Clearly, if $l = 0$, 1, 4, 5 or 7, then $q_{s_4 s_4} \ne -1$, and $\dim \mathfrak{B}(\mathcal{O}_{s_4}, \nu_l) = \infty$, by Lemma 1.3. The remained two cases correspond to $l = 2$, 6. We compute that $\mathcal{O}_{s_4} \cap M_{11}^{s_4} = \{s_4, s_4^{-1}\}$. It is easy to see that the braiding is negative.

CASE: $j = 6$ *or* 7. We compute that $M_{11}^{s_j} = \langle s_j \rangle \simeq \mathbb{Z}_8$, s_j^3 is in $\mathcal{O}_{s_j}$. Since 3 does not divide $|s_j| = 8$ we have that if $q_{s_j s_j} \ne -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, by Lemma 1.4. The remained case corresponds to $\rho(s_j) = \omega_8^4 = -1$, which satisfies $q_{s_j s_j} = -1$. We compute that $\mathcal{O}_{s_j} \cap M_{11}^{s_j} = \{s_j, s_j^3\}$. It is straightforward to prove that the braiding is negative.

CASE: $j = 10$. The element s_{10} is real and we compute that $M_{11}^{s_{10}} = \langle s_{10} \rangle \simeq \mathbb{Z}_6$. Now, we have that if $q_{s_{10} s_{10}} \ne -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_{10}}, \rho) = \infty$, by Lemma 1.3. The remained case corresponds to $\rho(s_{10}) = \omega_6^3 = -1$, which satisfies $q_{s_{10} s_{10}} = -1$. We compute that $\mathcal{O}_{s_{10}} \cap M_{11}^{s_{10}} = \{s_{10}, s_{10}^{-1}\}$. Then the braiding is negative.

CASE: $j = 5$. We compute that $M_{11}^{s_5}$ is a non-abelian group of order 48, whose character table is given by Table 3.

For every k, $1 \le k \le 8$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{11}^{s_5}$ whose character is μ_k. From Table 3, we can deduce that if $k \ne 4$, 5, 8, then $q_{s_5 s_5} \ne -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_5}, \rho_k) = \infty$, by Lemma 1.3.

On the other hand, if $k = 4$, 5 or 8, then $q_{s_5 s_5} = -1$. For these three cases we will prove that $\dim \mathfrak{B}(\mathcal{O}_{s_5}, \rho_k) = \infty$. First, we compute that $\mathcal{O}_{s_5} \cap M_{11}^{s_5}$ has 13 elements and it contains $\sigma_1 := s_5$, $\sigma_2 := (4,10)(5,8)(6,7)(9,11)$ and $\sigma_3 := (1,2)(5,7)(6,8)(9,11)$. Notice that these elements commute each other and $\sigma_2\sigma_3 = s_5$. Also, we compute that σ_2, $\sigma_3 \in \mathcal{O}_2^{M_{11}^{s_5}}$ – see Subsection 0.1. Now, we choose $g_1 := \mathrm{id}$,

$$g_2 := (1,9)(2,11)(4,10)(5,7) \quad \text{and} \quad g_3 := (1,2)(4,9)(6,7)(10,11).$$

These elements are in M_{11} and they satisfy

$$\sigma_1 g_1 = g_1 \sigma_1, \qquad \sigma_1 g_2 = g_2 \sigma_2, \qquad \sigma_1 g_3 = g_3 \sigma_3, \tag{2.1}$$

$$\sigma_2 g_1 = g_1 \sigma_2, \qquad \sigma_2 g_2 = g_2 \sigma_1, \qquad \sigma_2 g_3 = g_3 \sigma_2, \tag{2.2}$$

$$\sigma_3 g_1 = g_1 \sigma_3, \qquad \sigma_3 g_2 = g_2 \sigma_3, \qquad \sigma_3 g_3 = g_3 \sigma_1. \tag{2.3}$$

Assume that $k = 4$. Since σ_1, σ_2 and σ_3 commute there exists a basis $\{v_1, v_2\}$ of V_4, the vector space affording ρ_4, composed by simultaneous eigenvectors of $\rho_4(\sigma_1)$, $\rho_4(\sigma_2)$ and $\rho_4(\sigma_3)$. Let us say $\rho_4(\sigma_2)v_l = \lambda_l v_l$ and

k	1	2	3	4	5	6	7	8
$\lvert y_k \rvert$	1	2	3	2	6	4	8	8
$\lvert G^{y_k} \rvert$	48	4	6	48	6	8	8	8
$\lvert \mathcal{O}_{y_k} \rvert$	1	12	8	1	8	6	6	6
μ_1	1	1	1	1	1	1	1	1
μ_2	1	-1	1	1	1	1	-1	-1
μ_3	2	0	-1	2	-1	2	0	0
μ_4	2	0	-1	-2	1	0	$i\sqrt{2}$	-$i\sqrt{2}$
μ_5	2	0	-1	-2	1	0	-$i\sqrt{2}$	$i\sqrt{2}$
μ_6	3	-1	0	3	0	-1	1	1
μ_7	3	1	0	3	0	-1	-1	-1
μ_8	4	0	1	-4	-1	0	0	0

TABLE 3. Character table of $M_{11}^{s_5}$.

$\rho_4(\sigma_3)v_l = \kappa_l v_l$, $l = 1, 2$. Notice that λ_l, $\kappa_l = \pm 1$, $l = 1, 2$, due to $|\sigma_2| = 2 = |\sigma_3|$. Moreover, since $\sigma_2\sigma_3 = s_5$ we have that $\lambda_l\kappa_l = -1$, $l = 1$, 2. From Table 3, we can deduce that $\lambda_1 + \lambda_2 = 0$ because $\sigma_2 \in \mathcal{O}_2^{M_{11}^{s_5}}$ and $\mu_4(\mathcal{O}_2^{M_{11}^{s_5}}) = 0$. Reordering the basis we can suppose that $\lambda_1 = 1 = -\lambda_2$. We define $W := \mathbb{C}$ - span of $\{g_1v_1, g_2v_2, g_3v_2\}$. Hence, W is a braided vector subspace of $M(\mathcal{O}_{s_5}, \rho)$ of Cartan type. Indeed, it is straightforward to compute that the matrix of coefficients $\mathcal{Q}$ is

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}. \tag{2.4}$$

The corresponding Cartan matrix is given by

$$\mathcal{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{2.5}$$

By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_5}, \rho_4) = \infty$.

The case $k = 5$ is analogous to the case $k = 4$ because $\mu_5(\mathcal{O}_2^{M_{11}^{s_5}}) = 0$.

Finally, the case $k = 8$ can be reduced to the previous cases. Indeed, since σ_1, σ_2 and σ_3 commute there exists a basis $\{v_1, v_2, v_3, v_4\}$ of V_8, the vector space affording ρ_8, composed by simultaneous eigenvectors of $\rho_8(\sigma_1)$, $\rho_8(\sigma_2)$ and $\rho_8(\sigma_3)$. Let us say $\rho_8(\sigma_2)v_l = \lambda_l v_l$ and $\rho_8(\sigma_3)v_l = \kappa_l v_l$, $1 \leq l \leq 4$, where

λ_l, $\kappa_l = \pm 1$, $1 \le l \le 4$, due to $|\sigma_2| = 2 = |\sigma_3|$. From Table 3, we can deduce that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$ because $\mu_8(\mathcal{O}_2^{M_{11}^{s_5}}) = 0$. This implies that there exist r, $t \in \{1, 2, 3, 4\}$ such that $\lambda_r = 1 = -\lambda_t$. Now, if we define $W := \mathbb{C}$ - span of $\{g_1 v_r, g_2 v_t, g_3 v_t\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_5}, \rho)$ of Cartan type, with Cartan matrix given by (2.5). Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_5}, \rho_8) = \infty$. $\square$

Remark 2.2. The group $M_{11}^{s_8}$ has 9 conjugacy classes. So, we can point out the following fact: there are 84 possible pairs $(\mathcal{O}, \rho)$ for M_{11}; 79 of them have infinite-dimensional Nichols algebras, and 5 have negative braiding.

2.2. **The group** M_{12}**.** The Mathieu simple group M_{12} can be given as a subgroup of $\mathbb{S}_{12}$ in the following form

$$\begin{aligned} M_{12} := \langle\, &(1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)(4,10,5,6), \\ &(1,12)(2,11)(3,6)(4,8)(5,9)(7,10)\,\rangle. \end{aligned}$$

In Table 4, we show the character table of M_{12}, with $A = (-1 - i\sqrt{11})/2$. We take the following elements to be the representatives of the conjugacy classes of M_{12}: $s_1 := \mathrm{id}$,

$s_2 := (1,8,12)(2,3,11,9,10,6)(4,5)$, $s_3 := (1,12,8)(2,11,10)(3,9,6)$,

$s_4 := (2,9)(3,10)(4,5)(6,11)$, $s_5 := (1,12,7,4)(2,9,10,5,11,3,6,8)$,

$s_6 := (1,7)(2,10,11,6)(3,8,9,5)(4,12)$, $s_7 := (1,9,4,2,11,8)(3,10,12,5,6,7)$,

$s_8 := (1,4,11)(2,8,9)(3,12,6)(5,7,10)$, $s_9 := (1,2)(3,5)(4,8)(6,10)(7,12)(9,11)$,

$s_{10} := (1,7,2,6,5)(3,9,12,10,11)$, $s_{11} := (2,10,6,4,12,5,7,3,8,11,9)$,

$s_{12} := (2,6,12,7,8,9,10,4,5,3,11)$, $s_{13} := (1,2,7,10,5,6,8,12,9,4)(3,11)$,

$s_{14} := (1,10,11,8,4,5,12,3)(6,9)$, $s_{15} := (1,11,4,12)(3,10,8,5)$.

In the following statement, we summarize our study by mean of abelian subracks in the group M_{12}.

Theorem 2.3. *Let $\rho \in \widehat{M_{12}^{s_j}}$, with $1 \le j \le 15$. The braiding is negative in the cases $j = 2$, 5, 13 and 14, with $\rho = \chi_{(-1)}$. Otherwise,* $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$.

Proof. CASE: $j = 3$, 8, 10. From Table 4, we see that s_j is real. By Lemma 1.3, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{12}^{s_j}}$.

CASE: $j = 11$, 12. We compute that s_j^3 and s_j^9 are in $\mathcal{O}_{s_j}$, and $s_j^3 \neq s_j^9$. By Lemma 1.4, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{12}^{s_j}}$, since $|s_j| = 11$.

CASE: $j = 2$. The element s_2 is real and we compute that $M_{12}^{s_2} = \langle s_2 \rangle \simeq \mathbb{Z}_6$. Now, if $q_{s_2 s_2} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \rho) = \infty$, by Lemma 1.3. The

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert s_j\rvert$	1	6	3	2	8	4	6	3	2	5	11	11	10	8	4
$\lvert G^{s_j}\rvert$	95040	6	54	192	8	32	12	36	240	10	11	11	10	8	32
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	11	0	2	3	1	3	-1	-1	-1	1	0	0	-1	-1	-1
χ_3	11	0	2	3	-1	-1	-1	-1	-1	1	0	0	-1	1	3
χ_4	16	0	-2	0	0	0	1	1	4	1	A'	A	-1	0	0
χ_5	16	0	-2	0	0	0	1	1	4	1	A	A'	-1	0	0
χ_6	45	0	0	-3	-1	1	-1	3	5	0	1	1	0	-1	1
χ_7	54	0	0	6	0	2	0	0	6	-1	-1	-1	1	0	2
χ_8	55	1	1	7	-1	-1	1	1	-5	0	0	0	0	-1	-1
χ_9	55	-1	1	-1	-1	3	1	1	-5	0	0	0	0	1	-1
χ_{10}	55	-1	1	-1	1	-1	1	1	-5	0	0	0	0	-1	3
χ_{11}	66	-1	3	2	0	-2	0	0	6	1	0	0	1	0	-2
χ_{12}	99	0	0	3	1	-1	-1	3	-1	-1	0	0	-1	1	-1
χ_{13}	120	1	3	-8	0	0	0	0	0	0	-1	-1	0	0	0
χ_{14}	144	0	0	0	0	0	1	-3	4	-1	1	1	-1	0	0
χ_{15}	176	0	-4	0	0	0	-1	-1	-4	1	0	0	1	0	0

TABLE 4. Character table of M_{12}.

remained case corresponds to $\rho(s_2) = \omega_6^3 = -1$, which satisfies $q_{s_2 s_2} = -1$. We compute that $\mathcal{O}_{s_2} \cap M_{12}^{s_2} = \{s_2, s_2^{-1}\}$, and that the braiding is negative.

CASE: $j = 5$, 14. The element s_j is real and $M_{12}^{s_j} = \langle s_j \rangle \simeq \mathbb{Z}_8$. If $q_{s_j s_j} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, by Lemma 1.3. The remained case corresponds to $\rho(s_j) = \omega_8^4 = -1$, which satisfies $q_{s_j s_j} = -1$. We compute that $\mathcal{O}_{s_j} \cap M_{12}^{s_j} = \{s_j, s_j^3, s_j^5, s_j^7\}$, and that the braiding is negative.

CASE: $j = 13$. The element s_{13} is real and $M_{12}^{s_{13}} = \langle s_{13} \rangle \simeq \mathbb{Z}_{10}$. If $q_{s_{13} s_{13}} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_{13}}, \rho) = \infty$, by Lemma 1.3. The remained case corresponds to $\rho(s_{13}) = \omega_{10}^5 = -1$, which satisfies $q_{s_{13} s_{13}} = -1$. We compute that $\mathcal{O}_{s_{13}} \cap M_{12}^{s_{13}} = \{s_{13}, s_{13}^3, s_{13}^7, s_{13}^9\}$, and that the braiding is negative.

CASE: $j = 7$. The element s_7 is real and we compute that

$$M_{12}^{s_7} = \langle x, s_7 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_6,$$

with $x := (1,12)(2,7)(3,11)(4,6)(5,9)(8,10)$. Let us define $\{\nu_0, \dots, \nu_5\}$, where $\nu_l(s_7) := \omega_6^l$, $0 \leq l \leq 5$. So,

$$\widehat{M_{12}^{s_7}} = \{\epsilon \otimes \nu_l,\ \mathrm{sgn} \otimes \nu_l \mid 0 \leq l \leq 5\},$$

where ϵ and sgn mean the trivial and the sign representations of $\mathbb{Z}_2$, respectively. Clearly, if $\rho \in \widehat{M_{12}^{s_7}}$, with $l \neq 3$, then $q_{s_7 s_7} \neq -1$, and $\dim \mathfrak{B}(\mathcal{O}_{s_7}, \rho) = \infty$, by Lemma 1.3. The remained two cases are $\epsilon \otimes \nu_3$ and $\mathrm{sgn} \otimes \nu_3$. We will prove that also the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_7}, \rho)$ is infinite-dimensional. We compute that $\mathcal{O}_{s_7} \cap M_{12}^{s_7}$ has 6 elements and it contains $\sigma_1 := s_7$,

$$\sigma_2 := (1,5,4,7,11,10)(2,3,8,12,9,6), \quad \sigma_3 := (1,3,4,12,11,6)(2,5,8,7,9,10).$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lvert y_k\rvert$	1	4	4	2	4	8	4	2	2	4	4	4	8	4
$\lvert G^{y_k}\rvert$	32	8	16	32	16	8	16	16	8	32	32	16	8	16
$\lvert\mathcal{O}_{y_k}\rvert$	1	4	2	1	2	4	2	2	4	1	1	2	4	2
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	1	-1
μ_3	1	1	1	1	-1	-1	-1	1	1	1	1	-1	-1	-1
μ_4	1	-1	1	1	1	-1	1	1	-1	1	1	1	-1	1
μ_5	1	1	1	1	-i	-i	-i	-1	-1	-1	-1	i	i	i
μ_6	1	-1	1	1	-i	i	-i	-1	1	-1	-1	i	-i	i
μ_7	1	1	1	1	i	i	i	-1	-1	-1	-1	-i	-i	-i
μ_8	1	-1	1	1	i	-i	i	-1	1	-1	-1	-i	i	-i
μ_9	2	0	-2	2	0	0	0	-2	0	2	2	0	0	0
μ_{10}	2	0	-2	2	0	0	0	2	0	-2	-2	0	0	0
μ_{11}	2	0	0	-2	1+i	0	-1-i	0	0	-2i	2i	1-i	0	-1+i
μ_{12}	2	0	0	-2	-1-i	0	1+i	0	0	-2i	2i	-1+i	0	1-i
μ_{13}	2	0	0	-2	-1+i	0	1-i	0	0	2i	-2i	-1-i	0	1+i
μ_{14}	2	0	0	-2	1-i	0	-1+i	0	0	2i	-2i	1+i	0	-1-i

TABLE 5. Character table of $M_{12}^{s_6}$.

Notice that $\sigma_2\sigma_3 = s_7^{-1}$. We take $g_1 := \mathrm{id}$, $g_2 := (2,7,12)(3,9,5)(6,8,10)$ and $g_3 := g_2^{-1}$. We compute that the following relations hold

$$\sigma_1 g_1 = g_1\sigma_1, \qquad \sigma_1 g_2 = g_2\sigma_3, \qquad \sigma_1 g_3 = g_3\sigma_2, \tag{2.6}$$

$$\sigma_2 g_1 = g_1\sigma_2, \qquad \sigma_2 g_2 = g_2\sigma_1, \qquad \sigma_2 g_3 = g_3\sigma_3, \tag{2.7}$$

$$\sigma_3 g_1 = g_1\sigma_3, \qquad \sigma_3 g_2 = g_2\sigma_2, \qquad \sigma_3 g_3 = g_3\sigma_1. \tag{2.8}$$

If $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_7}, \rho)$ of Cartan type, with matrix of coefficients given by

$$\mathcal{Q}_1 := \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}, \qquad \mathcal{Q}_2 := \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \tag{2.9}$$

for $\rho = \epsilon \otimes \nu_3$ and $\rho = \mathrm{sgn} \otimes \nu_3$, respectively. In both cases the associated Cartan matrix is given by (2.5). By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_7}, \rho) = \infty$.

CASE: $j = 6$. The element s_6 is real and we compute that $M_{12}^{s_6}$ is a non-abelian group of order 32, whose character table is given by Table 5.

For every k, $1 \le k \le 14$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{12}^{s_6}$ whose character is μ_k. From Table 5, we can conclude that

if $k \neq 5$, 6, 7, 8, 10, then $q_{s_6 s_6} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho_k) = \infty$, by Lemma 1.3. On the other hand, if $k = 5$, 6, 7, 8 or 10, then $q_{s_6 s_6} = -1$. For these five cases we will prove that $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho_k) = \infty$. First, we compute that $\mathcal{O}_{s_6} \cap M_{12}^{s_6} = \{\sigma_l \mid 1 \leq l \leq 6\}$, where

$$\sigma_1 := (1,4,7,12)(2,6,11,10)(3,9)(5,8),$$
$$\sigma_2 := (1,4,7,12)(2,11)(3,5,9,8)(6,10),$$

$\sigma_3 := s_6^{-1}$, $\sigma_4 := s_6$, $\sigma_5 := \sigma_1^{-1}$ and $\sigma_6 := \sigma_2^{-1}$. We compute that these elements commute each other and that $\sigma_2 \in \mathcal{O}_7^{M_{12}^{s_6}}$ and σ_5, $\sigma_6 \in \mathcal{O}_{14}^{M_{12}^{s_6}}$. We take in M_{12} the following elements:

$$g_1 := (1,8,6,12,3,2)(4,9,11,7,5,10), \quad g_2 := (1,6,12,11,7,10,4,2)(5,8)$$

and $g_3 := (1,12)(4,7)(5,8)(6,10)$. Then $\sigma_r g_r = g_r s_6$, $1 \leq r \leq 3$, and

$$\sigma_2 g_1 = g_1 \sigma_5, \qquad \sigma_1 g_2 = g_2 \sigma_5, \qquad \sigma_1 g_3 = g_3 \sigma_5,$$
$$\sigma_3 g_1 = g_1 \sigma_6, \qquad \sigma_3 g_2 = g_2 \sigma_4, \qquad \sigma_2 g_3 = g_3 \sigma_6.$$

Assume that $k = 5$, 6, 7 or 8. We define $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3\}$. Hence, W is a braided vector subspace of $M(\mathcal{O}_{s_6}, \rho_k)$ of Cartan type. From Table 5, we can calculate that the matrix of coefficients $\mathcal{Q}$ is

$$\mathcal{Q}_3 := \begin{pmatrix} -1 & i & i \\ i & -1 & i \\ i & i & -1 \end{pmatrix}, \qquad \mathcal{Q}_4 := \begin{pmatrix} -1 & -i & -i \\ -i & -1 & -i \\ -i & -i & -1 \end{pmatrix}, \tag{2.10}$$

for $k = 5$ or 6, and $k = 7$ or 8, respectively. In all the cases the associated Cartan matrix is as in (2.5). Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho_k) = \infty$.

Assume that $k = 10$. Since σ_5 and σ_6 commute there exists a basis $\{v_1, v_2\}$ of V_{10}, the vector space affording ρ_{10}, composed by simultaneous eigenvectors of $\rho_{10}(\sigma_5)$ and $\rho_{10}(\sigma_6)$. Let us say $\rho_{10}(\sigma_5)v_l = \lambda_l v_l$ and $\rho_{10}(\sigma_6)v_l = \kappa_l v_l$, $l = 1$, 2. Notice that $\kappa_l, \lambda_l = \pm 1, \pm i$, due to $|\sigma_5| = 4 = |\sigma_6|$. From Table 5, we can deduce that $\lambda_1 + \lambda_2 = 0 = \kappa_1 + \kappa_2$ because σ_5, $\sigma_6 \in \mathcal{O}_{14}^{M_{12}^{s_6}}$ and $\mu_{10}(\mathcal{O}_{14}^{M_{12}^{s_6}}) = 0$. Also, since $\sigma_5 \sigma_6 = s_6^{-1}$ we have that $\lambda_l \kappa_l = -1$, $l = 1$, 2. Now, we consider the four possibilities: $\lambda_1 = \pm 1$, $\pm i$.

(i) If $\lambda_1 = \pm 1$, then we take $W := \mathbb{C}$ - span of $\{g_1 v_1, g_2 v_2, g_3 v_1\}$.

(ii) If $\lambda_1 = \pm i$, then we take $W := \mathbb{C}$ - span of $\{g_1 v_1, g_2 v_1, g_3 v_1\}$.

In both cases, W is a braided vector subspace of $M(\mathcal{O}_{s_6}, \rho_{10})$ of Cartan type. We calculate that the matrices of coefficients are given by $\mathcal{Q}_1$ (resp. $\mathcal{Q}_2$) for the case $\lambda_1 = 1$ (resp. $\lambda_1 = -1$) – see (2.9); whereas the matrices of coefficients are given by $\mathcal{Q}_3$ (resp. $\mathcal{Q}_4$) for the case $\lambda_1 = i$ (resp. $\lambda_1 = -i$) –

k	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert y_k\rvert$	1	2	3	8	6	4	2	2	4	8	4	2	4
$\lvert G^{y_k}\rvert$	192	8	6	8	6	32	192	32	16	8	32	16	16
$\lvert\mathcal{O}_{y_k}\rvert$	1	24	32	24	32	6	1	6	12	24	6	12	12
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	-1	1	-1	1	1	1	1	-1	-1	1	1	-1
μ_3	2	0	-1	0	-1	2	2	2	0	0	2	2	0
μ_4	3	-1	0	1	0	-1	3	-1	1	-1	3	-1	1
μ_5	3	1	0	-1	0	-1	3	-1	-1	1	3	-1	-1
μ_6	3	-1	0	-1	0	3	3	-1	1	1	-1	-1	1
μ_7	3	1	0	-1	0	-1	3	3	1	-1	-1	-1	1
μ_8	3	1	0	1	0	3	3	-1	-1	-1	-1	-1	-1
μ_9	3	-1	0	1	0	-1	3	3	-1	1	-1	-1	-1
μ_{10}	4	0	1	0	-1	0	-4	0	2	0	0	0	-2
μ_{11}	4	0	1	0	-1	0	-4	0	-2	0	0	0	2
μ_{12}	6	0	0	0	0	-2	6	-2	0	0	-2	2	0
μ_{13}	8	0	-1	0	1	0	-8	0	0	0	0	0	0

TABLE 6. Character table of $M_{12}^{s_4}$.

see (2.10). In all these cases, the associated Cartan matrix is given by (2.5). Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho_{10}) = \infty$.

CASE: $j = 15$. We compute that $M_{12}^{s_{15}} \simeq M_{12}^{s_6}$. This implies that this case is analogous to the case $j = 6$, since $\mathcal{O}_{s_{15}} \simeq \mathcal{O}_{s_6}$ as racks.

CASE: $j = 4$. We compute that $M_{12}^{s_4}$ is a non-abelian group of order 192, whose character table is given by Table 6.

For every k, $1 \leq k \leq 13$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{12}^{s_4}$ whose character is μ_k. From Table 6, we can conclude that if $k \neq 10$, 11, 13, then $q_{s_4 s_4} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_4}, \rho_k) = \infty$, by Lemma 1.3.

Assume that $k = 10$, 11 or 13; thus $q_{s_4 s_4} = -1$. We will prove that $\dim \mathfrak{B}(\mathcal{O}_{s_4}, \rho_k) = \infty$. First, we compute that $\mathcal{O}_{s_4} \cap M_{12}^{s_4}$ has 31 elements, and that it contains $\sigma_1 := s_4$,

$$\sigma_2 := (3,4)(5,10)(6,11)(7,12) \quad \text{and} \quad \sigma_3 := (2,9)(3,5)(4,10)(7,12).$$

We compute that σ_2 and σ_3 commute, $\sigma_3 \in \mathcal{O}_2^{M_{12}^{s_4}}$ and $\sigma_2\sigma_3 = s_4$. Now, we choose in M_{12} the following elements: $g_1 := \mathrm{id}$,

$$g_2 := (2,7)(3,5)(6,11)(9,12) \quad \text{and} \quad g_3 := (2,9)(5,10)(6,7)(11,12).$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\|y_k\|$	1	2	4	5	2	4	10	2	2	6	6	3	6	2
$\|G^{y_k}\|$	240	16	8	10	16	8	10	240	24	12	12	12	12	24
$\|\mathcal{O}_{y_k}\|$	1	15	30	24	15	30	24	1	10	20	20	20	20	10
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	-1
μ_3	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1
μ_4	1	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	-1
μ_5	4	0	0	-1	0	0	-1	4	-2	1	1	1	1	-2
μ_6	4	0	0	-1	0	0	1	-4	2	-1	-1	1	1	-2
μ_7	4	0	0	-1	0	0	-1	4	2	1	-1	1	-1	2
μ_8	4	0	0	-1	0	0	1	-4	-2	-1	1	1	-1	2
μ_9	5	1	-1	0	1	-1	0	5	1	-1	1	-1	1	1
μ_{10}	5	1	-1	0	-1	1	0	-5	-1	1	-1	-1	1	1
μ_{11}	5	1	1	0	1	1	0	5	-1	-1	-1	-1	-1	-1
μ_{12}	5	1	1	0	-1	-1	0	-5	1	1	1	-1	-1	-1
μ_{13}	6	-2	0	1	-2	0	1	6	0	0	0	0	0	0
μ_{14}	6	-2	0	1	2	0	-1	-6	0	0	0	0	0	0

TABLE 7. Character table of $M_{12}^{s_9}$.

It is easy to check that they satisfy the relations given in (2.1), (2.2) and (2.3). From Table 6, we have that $\mu_k(\mathcal{O}_2^{M_{12}^{s_4}}) = 0$. Now, we can proceed as in the case $j = 5$ and $k = 8$ of the proof of Theorem 2.1. Thus, we can obtain a braided vector subspace of $M(\mathcal{O}_{s_4}, \rho_k)$ of Cartan type whose associated Cartan matrix is not of finite type. Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_4}, \rho_k) = \infty$, for $k = 10$, 11, 13.

CASE: $j = 9$. We compute that $M_{12}^{s_9}$ is a non-abelian group of order 240, whose character table is given by Table 7.

For every k, $1 \leq k \leq 14$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{12}^{s_9}$ whose character is μ_k. From Table 7, we can conclude that if $k \neq 2$, 3, 6, 8, 10, 12, 14, then $q_{s_9 s_9} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_9}, \rho_k) = \infty$, by Lemma 1.3. On the other hand, if $k = 2$, 3, 6, 8, 10, 12 or 14, then $q_{s_9 s_9} = -1$. For these cases we will prove that $\dim \mathfrak{B}(\mathcal{O}_{s_9}, \rho_k) = \infty$. First, we compute that $\mathcal{O}_{s_9} \cap M_{12}^{s_9}$ has 36 elements, and that it contains $\sigma_1 := s_9$,

$$\sigma_2 := (1,3)(2,5)(4,6)(7,9)(8,10)(11,12),$$
$$\sigma_3 := (1,5)(2,3)(4,10)(6,8)(7,11)(9,12).$$

We set $g_1 := \mathrm{id}$, $g_2 := (2,3,4)(5,6,8)(7,11,9)$, $g_3 := (2,5,3)(4,8,6)(7,12,9)$; these elements are in M_{12} and they satisfy $\sigma_r g_r = g_r s_9$, $1 \leq r \leq 3$, $\sigma_3 g_2 = g_2\sigma_3$ and

$$\begin{aligned} &\sigma_2 g_1 = g_1\sigma_2, && \sigma_1 g_2 = g_2\gamma_{1,2}, && \sigma_1 g_3 = g_3\sigma_2,\\ &\sigma_3 g_1 = g_1\sigma_3, && \sigma_3 g_2 = g_2\gamma_{3,2}, && \sigma_2 g_3 = g_3\sigma_3, \end{aligned}$$

where $\gamma_{1,2} = (1,4)(2,8)(3,6)(5,10)(7,11)(9,12)$ and $\gamma_{3,2} = s_9\gamma_{1,2}^{-1}$. Also, we compute that $\sigma_2\sigma_3 = s_9 = \sigma_3\sigma_2$, and that σ_3, $\gamma_{1,2} \in \mathcal{O}_{14}^{M_{12}^{s_9}}$ and $\gamma_{3,2} \in \mathcal{O}_{9}^{M_{12}^{s_9}}$.

Assume that $k = 2$ or 3. Let us define $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3\}$. Hence, W is a braided vector subspace of $M(\mathcal{O}_{s_9}, \rho_k)$ of Cartan type. From Table 7, it is straightforward to calculate that the associated Cartan matrix is as in (2.5). Thus, $\dim \mathfrak{B}(\mathcal{O}_{s_9}, \rho_k) = \infty$.

Assume that $k = 6$. Since σ_1, σ_2, σ_3, $\gamma_{1,2}$ and $\gamma_{3,2}$ commute there exists a basis $\{v_1, v_2, v_3, v_4\}$ of V_6, the vector space affording ρ_6, composed by simultaneous eigenvectors of $\rho_6(\sigma_1) = -\mathrm{Id}$, $\rho_6(\sigma_2)$, $\rho_6(\sigma_3)$ $\rho_6(\gamma_{1,2})$ and $\rho_6(\gamma_{3,2})$. Let us call $\rho_6(\sigma_2)v_l = \lambda_l v_l$, $\rho_6(\sigma_3)v_l = \kappa_l v_l$, $1 \leq l \leq 4$, where λ_l, $\kappa_l = \pm 1$, $1 \leq l \leq 4$, due to $|\sigma_2| = |\sigma_3| = 2$. From Table 7, we have that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -2 = -\kappa_1 - \kappa_2 - \kappa_3 - \kappa_4$. So, we have that λ_1, λ_2, λ_3 and λ_4 are not all equal to 1 or -1. On the other hand, since $\sigma_2\sigma_3 = s_9$ and $q_{s_9 s_9} = -1$, we have that $\lambda_l\kappa_l = -1$, $1 \leq l \leq 4$. Now, if $W := \mathbb{C}$ - span of $\{g_1v_l, g_2v_l, g_3v_l \,|\, 1 \leq l \leq 4\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_9}, \rho_6)$ of Cartan type, whose associated Cartan matrix $\mathcal{A}$ has at least two row with three -1 or more. This means that the corresponding Dynkin diagram has at least two vertices with three edges or more; thus, $\mathcal{A}$ is not of finite type. Hence, $\dim \mathfrak{B}(\mathcal{O}_{s_9}, \rho_6) = \infty$.

For the cases $k = 8$, 9, 10, 12 or 14, we proceed in an analogous way. $\square$

Remark 2.4. We compute that the groups $M_{12}^{s_3}$, $M_{12}^{s_8}$ and $M_{12}^{s_{10}}$, have 10, 12 and 10 conjugacy classes, respectively. Hence, there are 168 possible pairs $(\mathcal{O}, \rho)$ for M_{12}. We conclude that 164 of them lead to infinite-dimensional Nichols algebras, and 4 have negative braiding.

2.3. **The group M_{22}.** The Mathieu simple group M_{22} can be given as a subgroup of $\mathbb{S}_{22}$ in the following form

$$\begin{aligned} M_{22} := \langle &(1,2,3,4,5,6,7,8,9,10,11)(12,13,14,15,16,17,18,19,20,21,22),\\ &(1,4,5,9,3)(2,8,10,7,6)(12,15,16,20,14)(13,19,21,18,17),\\ &(1,21)(2,10,8,6)(3,13,4,17)(5,19,9,18)(11,22)(12,14,16,20)\,\rangle. \end{aligned}$$

In Table 8, we show the character table of M_{22}, with $A = (-1 - i\sqrt{11})/2$ and $C = (-1 - i\sqrt{7})/2$. The representatives of the conjugacy classes of M_{22}

j	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert s_j\rvert$	1	4	2	8	7	7	5	11	11	4	6	3
$\lvert G^{s_j}\rvert$	443520	32	384	8	7	7	5	11	11	16	12	36
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	21	1	5	-1	0	0	1	-1	-1	1	-1	3
χ_3	45	1	-3	-1	C	C'	0	1	1	1	0	0
χ_4	45	1	-3	-1	C'	C	0	1	1	1	0	0
χ_5	55	3	7	1	-1	-1	0	0	0	-1	1	1
χ_6	99	3	3	-1	1	1	-1	0	0	-1	0	0
χ_7	154	-2	10	0	0	0	-1	0	0	2	1	1
χ_8	210	-2	2	0	0	0	0	1	1	-2	-1	3
χ_9	231	-1	7	-1	0	0	1	0	0	-1	1	-3
χ_{10}	280	0	-8	0	0	0	0	A'	A	0	1	1
χ_{11}	280	0	-8	0	0	0	0	A	A'	0	1	1
χ_{12}	385	1	1	1	0	0	0	0	0	1	-2	-2

TABLE 8. Character table of M_{22}.

are $s_1 := \mathrm{id}$,

$$
\begin{aligned}
s_2 &:= (1,10,13,17)(2,3,14,15)(4,20,18,7)(5,21)(6,22)(9,11,12,16),\\
s_3 &:= (1,13)(2,14)(3,15)(4,18)(7,20)(9,12)(10,17)(11,16),\\
s_4 &:= (1,8,17,5,11,15,3,7)(2,14,9,16)(4,20)(6,21,13,22,19,18,12,10),\\
s_5 &:= (1,12,16,15,19,11,18)(2,7,9,14,13,10,6)(3,22,4,17,5,21,8),\\
s_6 &:= (1,15,18,16,11,12,19)(2,14,6,9,10,7,13)(3,17,8,4,21,22,5),\\
s_7 &:= (1,4,2,6,3)(5,15,12,22,18)(7,8,11,19,20)(9,17,10,14,21),\\
s_8 &:= (1,18,4,12,15,8,3,17,19,7,6)(2,9,16,11,13,22,20,5,10,14,21),\\
s_9 &:= (1,4,15,3,19,6,18,12,8,17,7)(2,16,13,20,10,21,9,11,22,5,14),\\
s_{10} &:= (1,6,5,17)(3,8)(4,11)(7,13,16,14)(9,12,22,15)(10,20,18,19),\\
s_{11} &:= (1,7,22)(2,13,6,14,5,3)(4,10)(8,16,9,20,19,17)(11,15,21)(12,18),\\
s_{12} &:= (1,22,7)(2,6,5)(3,13,14)(8,9,19)(11,21,15)(16,20,17).
\end{aligned}
$$

In the following statement, we summarize our study by mean of abelian subracks in the group M_{22}.

Theorem 2.5. *Let $\rho \in \widehat{M_{22}^{s_j}}$, with $1 \le j \le 12$. If $j = 4$ and $\rho = \chi_{(-1)}$, then the braiding is negative. Otherwise,* $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$.

Proof. CASE: $j = 7$, 12. From Table 8, we see that s_j is real. By Lemma 1.3, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{22}^{s_j}}$.

CASE: $j = 5$, 6. We compute that s_j^2 and s_j^4 are in $\mathcal{O}_{s_j}$. Since $|s_j| = 7$ we have that $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{22}^{s_j}}$, by Lemma 1.4.

CASE: $j = 8$, 9. We compute that s_j^3 and s_j^9 are in $\mathcal{O}_{s_j}$. Since $|s_j| = 11$ we have that $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{22}^{s_j}}$, by Lemma 1.4.

CASE: $j = 4$. We compute that $M_{22}^{s_4} = \langle s_4 \rangle \simeq \mathbb{Z}_8$. From Table 8, we see that s_4 is real. Thus, if $q_{s_j s_j} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, by Lemma 1.3. The remained case corresponds to $\rho(s_4) = \omega_8^4 = -1$, which satisfies $q_{s_j s_j} = -1$. We compute that $\mathcal{O}_{s_4} \cap M_{22}^{s_4} = \{s_4, s_4^3, s_4^5, s_4^7\}$. It is straightforward to prove that the braiding is negative.

CASE: $j = 11$. We compute that $M_{22}^{s_{11}} = \langle x, s_{11} \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$, where

$$x := (2,9)(3,16)(4,12)(5,8)(6,19)(10,18)(13,20)(14,17).$$

Let us define $\{\nu_0, \dots, \nu_5\}$, where $\nu_l(s_{11}) := \omega_6^l$, $0 \leq l \leq 5$. So,

$$\widehat{M_{22}^{s_{11}}} = \{\epsilon \otimes \nu_l, \operatorname{sgn} \otimes \nu_l \mid 0 \leq l \leq 5\},$$

where ϵ and sgn mean the trivial and the sign representations of $\mathbb{Z}_2$, respectively. Since s_{11} is real we have that if $\rho \in \widehat{M_{22}^{s_{11}}}$, with $l \neq 3$, then $q_{s_{11}s_{11}} \neq -1$, and $\dim \mathfrak{B}(\mathcal{O}_{s_{11}}, \rho) = \infty$, by Lemma 1.3. The remained two cases are $\rho = \epsilon \otimes \nu_3$ and $\rho = \operatorname{sgn} \otimes \nu_3$. We will prove that also the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_{11}}, \rho)$ is infinite-dimensional. First, we compute that $\mathcal{O}_{s_{11}} \cap M_{22}^{s_{11}}$ contains $\sigma_1 := s_{11}$,

$$\sigma_2 := (1,7,22)(2,8,6,9,5,19)(3,17,13,16,14,20)(4,12)(10,18)(11,15,21),$$
$$\sigma_3 := (1,7,22)(2,20,6,17,5,16)(3,9,13,19,14,8)(4,18)(10,12)(11,15,21).$$

We compute that $\sigma_2 = x s_{11}^4$ and $\sigma_3 = x s_{11}$. We choose $g_1 := \mathrm{id}$,

$$g_2 := (1,7,22)(3,19,16)(4,12,10)(8,20,13)(9,17,14)(11,21,15)$$

and $g_3 := g_2^{-1}$. These elements are in M_{22} and they satisfy the same relations as in (2.6), (2.7) and (2.8). Now, if $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_{11}}, \rho)$ of Cartan type with matrix of coefficients given by $\mathcal{Q}_1$ (resp. $\mathcal{Q}_2$) for the case $\rho = \epsilon \otimes \nu_3$ (resp. $\rho = \operatorname{sgn} \otimes \nu_3$) – see (2.9). In both cases the associated Cartan matrix is as in (2.5). By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_{11}}, \rho) = \infty$.

CASE: $j = 10$. We compute that $M_{22}^{s_{10}} = \langle x, s_{10} \rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$, where

$$x := (1,9,13,10)(3,11)(4,8)(5,22,14,18)(6,12,16,20)(7,19,17,15).$$

If we set $\{\nu_0, \dots, \nu_3\}$, where $\nu_l(-) := \omega_4^l$, $0 \le l \le 3$, then

$$\widehat{M_{22}^{s_{10}}} = \{\nu_l \otimes \nu_t, \ \mid 0 \le l, t \le 3\}.$$

If $\rho = \nu_l \otimes \nu_t$, with $t \neq 2$ and $0 \le l \le 3$, then $q_{s_{10}s_{10}} = (\nu_l \otimes \nu_t)(s_{10}) = \omega_4^t \neq -1$; since s_{10} is real we have that $\dim \mathfrak{B}(\mathcal{O}_{s_{10}}, \rho) = \infty$. Assume that $\rho = \nu_l \otimes \nu_2$, $0 \le l \le 3$. Let us define $\sigma_1 := s_{10}$, $\sigma_2 := s_{10}^{-1}$, $\sigma_3 := x$ and $\sigma_4 := x^{-1}$. We compute that $x \in \mathcal{O}_{s_{10}}$. Now, we choose $g_1 := \mathrm{id}$,

$$g_2 := (3,8)(4,11)(6,17)(7,16)(9,18)(10,22)(12,20)(15,19),$$
$$g_3 := (3,8,4)(5,13,14)(6,9,19)(7,22,15)(10,12,17)(16,18,20)$$

and $g_4 := g_3 g_2$. These elements are in M_{22} and satisfy that $\sigma_r g_r = g_r s_{10}$, $\sigma_r g_1 = g_1 \sigma_r$, $1 \le r \le 4$, and

$$\begin{aligned} \sigma_1 g_2 &= g_2\ s_{10}^{-1}, & \sigma_1 g_3 &= g_3\ x^{-1} s_{10}^{-1}, & \sigma_1 g_4 &= g_4\ x s_{10}^{-1}, \\ \sigma_3 g_2 &= g_2\ x^{-1} s_{10}^{2}, & \sigma_2 g_3 &= g_3\ x s_{10}, & \sigma_2 g_4 &= g_4\ x^{-1} s_{10}, \\ \sigma_4 g_2 &= g_2\ x s_{10}^{2}, & \sigma_4 g_3 &= g_3\ s_{10}^{-1}, & \sigma_3 g_4 &= g_4\ s_{10}^{-1}. \end{aligned}$$

If we define $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3, g_4\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_{10}}, \rho)$ of Cartan type, whose associated Cartan matrix is given by

$$\mathcal{A} = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}. \tag{2.11}$$

By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_{10}}, \rho) = \infty$.

CASE: $j = 2$. We have that s_2 is a real element, it has order 4 and we compute that $M_{22}^{s_2}$ is a non-abelian group of order 32.

Let $\rho = (\rho, V) \in \widehat{M_{22}^{s_2}}$. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_2}, \rho)$ is infinite-dimensional. If $q_{s_2s_2} \neq -1$, then the result follows from Lemma 1.3. Assume that $q_{s_2s_2} = -1$. We compute that $\mathcal{O}_{s_2} \cap M_{22}^{s_2}$ has 16 elements and it contains $\sigma_1 := s_2$,

$$\sigma_2 := (1,7,15,11)(2,12,10,4)(3,16,13,20)(5,6)(9,17,18,14)(21,22),$$
$$\sigma_3 := (1,9,3,4)(2,7,17,16)(5,22)(6,21)(10,11,14,20)(12,15,18,13).$$

We compute that σ_1, σ_2 and σ_3 commute and that $\sigma_2\sigma_3 = s_2^{-1}$. We choose $g_1 := \mathrm{id}$, $g_2 := (2,16,12)(3,13,15)(4,17,11)(5,6,21)(7,9,10)(14,20,18)$ and $g_3 := g_2^{-1}$. These elements belong to M_{22} and they satisfy the relations given by (2.6), (2.7) and (2.8). Now, we define $W := \mathbb{C}$ - span of $\{g_1 v, g_2 v, g_3 v\}$,

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\|y_k\|$	1	2	2	3	4	2	6	4	4	4	8	6	4	4	6	2	2
$\|G^{y_k}\|$	384	16	32	12	8	64	12	16	32	16	8	12	16	16	12	48	384
$\|\mathcal{O}_{y_k}\|$	1	24	12	32	48	6	32	24	12	24	48	32	24	24	32	8	1
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1
μ_3	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1
μ_4	1	-1	1	1	-1	1	1	-1	1	1	-1	1	-1	-1	1	1	1
μ_5	2	0	2	-1	0	2	-1	0	2	-2	0	1	0	0	1	-2	2
μ_6	2	0	2	-1	0	2	-1	0	2	2	0	-1	0	0	-1	2	2
μ_7	3	-1	-1	0	1	3	0	-1	-1	1	-1	0	1	1	0	-3	3
μ_8	3	1	-1	0	-1	3	0	1	-1	1	1	0	-1	-1	0	-3	3
μ_9	3	-1	-1	0	1	3	0	-1	-1	-1	1	0	-1	-1	0	3	3
μ_{10}	3	1	-1	0	-1	3	0	1	-1	-1	-1	0	1	1	0	3	3
μ_{11}	6	0	-2	0	0	-2	0	0	2	0	0	0	2	-2	0	0	6
μ_{12}	6	0	-2	0	0	-2	0	0	2	0	0	0	-2	2	0	0	6
μ_{13}	6	-2	2	0	0	-2	0	2	-2	0	0	0	0	0	0	0	6
μ_{14}	6	2	2	0	0	-2	0	-2	-2	0	0	0	0	0	0	0	6
μ_{15}	8	0	0	2	0	0	-2	0	0	0	0	0	0	0	0	0	-8
μ_{16}	8	0	0	-1	0	0	1	0	0	0	0	d	0	0	-d	0	-8
μ_{17}	8	0	0	-1	0	0	1	0	0	0	0	-d	0	0	d	0	-8

TABLE 9. Character table of $M_{22}^{s_3}$.

where $v \in V - 0$. Hence, it is straightforward to check that W is a braided vector subspace of $M(\mathcal{O}_{s_2}, \rho)$ of Cartan type whose associated Cartan matrix is given by (2.5). By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \rho) = \infty$.

CASE: $j = 3$. We compute that $M_{22}^{s_3}$ is a non-abelian group of order 384, whose character table is given by Table 9, where $d := i\sqrt{3}$.

For every k, $1 \leq k \leq 17$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{22}^{s_3}$ whose character is μ_k. From Table 9 and the fact that s_3 is real, we have that if $k \neq 15$, 16, 17, then $q_{s_3 s_3} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_3}, \rho_k) = \infty$, by Lemma 1.3. On the other hand, if $k = 15$, 16 or 17, then $q_{s_3 s_3} = -1$. For these cases we will prove that $\dim \mathfrak{B}(\mathcal{O}_{s_3}, \rho_k) = \infty$. First, we compute that $\mathcal{O}_{s_3} \cap M_{22}^{s_3}$ has 51 elements, and that it contains $\sigma_1 = s_3$,

$$\sigma_2 := (3,15)(4,20)(6,22)(7,18)(8,19)(9,16)(10,17)(11,12),$$
$$\sigma_3 := (1,13)(2,14)(4,7)(6,22)(8,19)(9,11)(12,16)(18,20).$$

We compute that σ_1, σ_2 and σ_3 commute each other, $\sigma_2\sigma_3 = s_3$ and that $\sigma_2 \in \mathcal{O}_3^{M_{22}^{s_3}}$. We choose in M_{22} the following elements: $g_1 := \mathrm{id}$,

$$g_2 := (1,6)(2,8)(4,11)(7,9)(12,18)(13,22)(14,19)(16,20),$$
$$g_3 := (3,6)(4,16)(7,11)(8,17)(9,20)(10,19)(12,18)(15,22).$$

They satisfy the same relations as in (2.1), (2.2) and (2.3).

Assume that $k = 15$, 16 or 17. Since σ_1, σ_2 and σ_3 commute there exists a basis $\{v_l \,|\, 1 \le l \le 8\}$ of V_k, the vector space affording ρ_k, composed by simultaneous eigenvectors of $\rho_k(\sigma_1) = -\operatorname{Id}$, $\rho_k(\sigma_2)$ and $\rho_k(\sigma_3)$. Let us say $\rho_k(\sigma_2)v_l = \lambda_l v_l$ and $\rho_k(\sigma_3)v_l = \kappa_l v_l$, $1 \le l \le 8$. Notice that λ_l, $\kappa_l = \pm 1$, $1 \le l \le 8$, due to $|\sigma_2| = 2 = |\sigma_3|$. On the other hand, since $\sigma_2\sigma_3 = s_3$ we have that $\lambda_l\kappa_l = -1$, $1 \le l \le 8$. From Table 9, we can deduce that $\sum_{l=1}^{8} \lambda_l = 0$ because $\mu_k(\mathcal{O}_3^{M_{22}^{s_3}}) = 0$. Reordering the basis we can suppose that $\lambda_1 = 1$ and $\lambda_2 = -1$; thus, $\kappa_1 = -1$ and $\kappa_2 = 1$. We define $W := \mathbb{C}$ - span of $\{g_1v_1, g_2v_2, g_3v_2\}$. Hence, W is a braided vector subspace of $M(\mathcal{O}_{s_3}, \rho_k)$ of Cartan type whose Cartan matrix is as in (2.5), and $\dim \mathfrak{B}(\mathcal{O}_{s_3}, \rho_k) = \infty$. $\square$

Remark 2.6. We compute that the groups $M_{22}^{s_2}$ and $M_{22}^{s_{12}}$ have 14 and 12 conjugacy classes, respectively. Hence, there are 132 possible pairs $(\mathcal{O}, \rho)$ for M_{22}; only one of then has negative braiding. The other pairs have infinite-dimensional Nichols algebras.

2.4. **The group M_{23}.** The Mathieu simple group M_{23} can be given as the subgroup of $\mathbb{S}_{23}$ generated by α_1 and α_2, where

$$\alpha_1 := (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23),$$
$$\alpha_2 := (3, 17, 10, 7, 9)(4, 13, 14, 19, 5)(8, 18, 11, 12, 23)(15, 20, 22, 21, 16).$$

The order of M_{23} is 10200960. In Table 10, we show the character table of M_{23}, where $A = (-1+i\sqrt{7}))/2$, $B = (-1+i\sqrt{11})/2$, $C = (-1+i\sqrt{15})/2$ and $D = (-1 + i\sqrt{23})/2$. We will denote the representatives of the conjugacy classes of M_{23} by s_j, $1 \le j \le 17$.

In the following statement, we summarize our study by mean of abelian subracks in the group M_{23}.

Theorem 2.7. *Let $\rho \in \widehat{M_{23}^{s_j}}$, with $1 \le j \le 17$. The braiding is negative in the cases $j = 9$, 12 and 13, with $\rho = \chi_{(-1)}$. Otherwise,* $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$.

Proof. CASE: $j = 3$, 5. From Table 10, we see that s_j is real. By Lemma 1.3, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{23}^{s_j}}$.

CASE: $j = 7$, 8, 14, 15, 16, 17 . We compute that s_j^2 and s_j^4 are in $\mathcal{O}_{s_j}$. Since $|s_j|$ is odd we have that $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{23}^{s_j}}$, by Lemma 1.4.

CASE: $j = 10$, 11. We compute that s_j^3 and s_j^9 are in $\mathcal{O}_{s_j}$. Since $|s_j| = 11$ we have that $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{23}^{s_j}}$, by Lemma 1.4.

CASE: $j = 12$, 13. We compute that $M_{23}^{s_j} = \langle s_j \rangle \simeq \mathbb{Z}_{14}$, and that s_j^9 and s_j^{11} are in $\mathcal{O}_{s_j}$. Thus, if $q_{s_js_j} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, by Lemma

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\|s_j\|$	1	2	3	4	5	6	7	7	8	11	11	14	14	15	15	23	23
$\|G^{s_j}\|$	$\|M_{23}\|$	2688	180	32	15	12	14	14	8	11	11	14	14	15	15	23	23
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	4	2	2	0	1	1	0	0	0	-1	-1	-1	-1	-1	-1
χ_3	45	-3	0	1	0	0	A	A'	-1	1	1	-A	-A'	0	0	-1	-1
χ_4	45	-3	0	1	0	0	A'	A	-1	1	1	-A'	-A	0	0	-1	-1
χ_5	230	22	5	2	0	1	-1	-1	0	-1	-1	1	1	0	0	0	0
χ_6	231	7	6	-1	1	-2	0	0	-1	0	0	0	0	1	1	1	1
χ_7	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	C	C'	1	1
χ_8	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	C'	C	1	1
χ_9	253	13	1	1	-2	1	1	1	-1	0	0	-1	-1	1	1	0	0
χ_{10}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	D	D'
χ_{11}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	D'	D
χ_{12}	896	0	-4	0	1	0	0	0	0	B	B'	0	0	1	1	-1	-1
χ_{13}	896	0	-4	0	1	0	0	0	0	B'	B	0	0	1	1	-1	-1
χ_{14}	990	-18	0	2	0	0	A	A'	0	0	0	A	A'	0	0	1	1
χ_{15}	990	-18	0	2	0	0	A'	A	0	0	0	A'	A	0	0	1	1
χ_{16}	1035	27	0	-1	0	0	-1	-1	1	1	1	-1	-1	0	0	0	0
χ_{17}	2024	8	-1	0	-1	-1	1	1	0	0	0	1	1	-1	-1	0	0

TABLE 10. Character table of M_{23}.

1.4. The remained case corresponds to $\rho(s_j) = \omega_{14}^7 = -1$, which satisfies $q_{s_j s_j} = -1$. We compute that $\mathcal{O}_{s_j} \cap M_{23}^{s_j} = \{s_j, s_j^9, s_j^{11}\}$. It is straightforward to prove that the braiding is negative.

CASE: $j = 9$. We compute that $M_{23}^{s_9} = \langle s_9 \rangle \simeq \mathbb{Z}_8$, and that s_9 is real. Thus, if $q_{s_9 s_9} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_9}, \rho) = \infty$, by Lemma 1.3. The remained case corresponds to $\rho(s_9) = \omega_8^4 = -1$, which satisfies $q_{s_9 s_9} = -1$. We compute that $\mathcal{O}_{s_9} \cap M_{23}^{s_9} = \{s_9, s_9^3, s_9^5, s_9^7\}$. It is easy to check that the braiding is negative.

CASE: $j = 6$. The representative is

$$s_6 = (1,19,20)(2,9,18,17,14,5)(3,21)(4,13,23,10,11,22)(6,8,15)(7,16),$$

it is real and has order 6. We compute that $M_{23}^{s_6} = \langle x, s_6 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$, with $x := (2,22)(3,7)(4,9)(5,11)(10,14)(13,18)(16,21)(17,23)$. Let us define $\{\nu_0, \dots, \nu_5\}$, where $\nu_l(s_6) := \omega_6^l$, $0 \le l \le 5$. So,

$$\widehat{M_{23}^{s_6}} = \{\epsilon \otimes \nu_l, \operatorname{sgn} \otimes \nu_l \mid 0 \le l \le 5\},$$

where ϵ and sgn mean the trivial and the sign representations of $\mathbb{Z}_2$, respectively. If $\rho \in \widehat{M_{23}^{s_6}}$, with $l \neq 3$, then $q_{s_6 s_6} \neq -1$, and $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho) = \infty$, by Lemma 1.3. The remained two cases are $\rho = \epsilon \otimes \nu_3$ and $\rho = \operatorname{sgn} \otimes \nu_3$. We will prove that also the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_6}, \rho)$ is infinite-dimensional. First,

we compute that $\mathcal{O}_{s_6} \cap M_{23}^{s_6}$ has 6 elements, and it contains $\sigma_1 := s_6$,

$$\sigma_2 := (1,19,20)(2,4,18,23,14,11)(3,16)(5,22,9,13,17,10)(6,8,15)(7,21),$$

$$\sigma_3 := (1,19,20)(2,10,18,22,14,13)(3,7)(4,5,23,9,11,17)(6,8,15)(16,21).$$

Also, we compute that $\sigma_2 = xs_6$ and $\sigma_3 = xs_6^4$. We choose $g_1 := \mathrm{id}$,

$$g_2 := (1,20,19)(3,21,7)(4,10,9)(5,11,13)(6,8,15)(17,23,22),$$

and $g_3 := g_2^{-1}$. These elements are in M_{23} and they satisfy the same relations as in (2.6), (2.7) and (2.8). If $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_6}, \rho)$ of Cartan type whose Cartan matrix is as in (2.5). Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho) = \infty$.

CASE: $j = 4$. The representative is

$$s_4 = (1,17,10,4)(2,8)(3,6,14,11)(5,12,13,21)(7,15)(16,20,22,23),$$

which has order 4 and it is real. We compute that $M_{23}^{s_4}$ is a non-abelian group of order 32. Let $\rho = (\rho, V) \in \widehat{M_{23}^{s_4}}$. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_4}, \rho)$ is infinite-dimensional. If $q_{s_4 s_4} \neq -1$, then the result follows from Lemma 1.3. Assume that $q_{s_4 s_4} = -1$. We compute that $\mathcal{O}_{s_4} \cap M_{23}^{s_4}$ has 16 elements and it contains $\sigma_1 := s_4$,

$$\sigma_2 := (1,12,6,20)(2,7)(3,16,4,5)(8,15)(10,21,11,23)(13,14,22,17),$$

$$\sigma_3 := (1,16,11,13)(2,15)(3,21,17,20)(4,23,14,12)(5,10,22,6)(7,8).$$

These elements commute and $\sigma_2\sigma_3 = s_4^{-1}$. We choose $g_1 := \mathrm{id}$,

$$g_2 := (2,15,7)(3,21,5)(4,20,13)(6,11,10)(12,16,17)(14,23,22)$$

and $g_3 := g_2^{-1}$. These elements belong to M_{23} and they satisfy the relations given by (2.6), (2.7) and (2.8). Now, we define $W := \mathbb{C}$ - span of $\{g_1 v, g_2 v, g_3 v\}$, where $v \in V - 0$. Hence, it is straightforward to check that W is a braided vector subspace of $M(\mathcal{O}_{s_4}, \rho)$ of Cartan type whose associated Cartan matrix is given by (2.5). By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_4}, \rho) = \infty$.

CASE: $j = 2$. The representative is

$$s_2 = (1,10)(3,14)(4,17)(5,13)(6,11)(12,21)(16,22)(20,23).$$

We compute that $M_{23}^{s_2}$ is a non-abelian group of order 2688, whose character table is given by Table 11, where $A = (-1 + i\sqrt{7})/2$ and $B = i\sqrt{3}$.

For every k, $1 \leq k \leq 16$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{23}^{s_2}$ whose character is μ_k. From Table 11, we have that if $k \neq 9$, 10, 11, 15, 16, then $q_{s_2 s_2} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \rho_k) = \infty$, by Lemma 1.3. On the other hand, if $k = 9$, 10, 11, 15 or 16, then $q_{s_2 s_2} = -1$. For these cases

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\|y_k\|$	1	7	7	14	14	3	6	6	6	2	4	8	2	4	2	4
$\|G^{y_k}\|$	2688	14	14	14	14	12	12	12	12	2688	32	8	192	16	32	8
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	3	A	A'	A	A'	0	0	0	0	3	-1	1	3	-1	-1	1
μ_3	3	A'	A	A'	A	0	0	0	0	3	-1	1	3	-1	-1	1
μ_4	6	-1	-1	-1	-1	0	0	0	0	6	2	0	6	2	2	0
μ_5	7	0	0	0	0	1	1	-1	-1	7	3	1	-1	-1	-1	-1
μ_6	7	0	0	0	0	1	1	1	1	7	-1	-1	7	-1	-1	-1
μ_7	7	0	0	0	0	1	1	-1	-1	7	-1	-1	-1	-1	3	1
μ_8	8	1	1	1	1	-1	-1	-1	-1	8	0	0	8	0	0	0
μ_9	8	1	1	-1	-1	2	-2	0	0	-8	0	0	0	0	0	0
μ_{10}	8	1	1	-1	-1	-1	1	B	-B	-8	0	0	0	0	0	0
μ_{11}	8	1	1	-1	-1	-1	1	-B	B	-8	0	0	0	0	0	0
μ_{12}	14	0	0	0	0	-1	-1	1	1	14	2	0	-2	-2	2	0
μ_{13}	21	0	0	0	0	0	0	0	0	21	1	-1	-3	1	-3	1
μ_{14}	21	0	0	0	0	0	0	0	0	21	-3	1	-3	1	1	-1
μ_{15}	24	A'	A	-A'	-A	0	0	0	0	-24	0	0	0	0	0	0
μ_{16}	24	A	A'	-A	-A'	0	0	0	0	-24	0	0	0	0	0	0

TABLE 11. Character table of $M_{23}^{s_2}$.

we will prove that $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \rho_k) = \infty$. First, we compute that $\mathcal{O}_{s_2} \cap M_{23}^{s_2}$ has 99 elements and it contains $\sigma_1 := s_2$ and

$$\sigma_2 := (3,6)(5,20)(7,9)(11,14)(12,21)(13,23)(15,18)(16,22).$$

We compute that $\sigma_2 \in \mathcal{O}_{15}^{M_{23}^{s_2}}$. Now, we choose $g_1 := \mathrm{id}$ and

$$g_2 := (1,7)(3,20)(4,18)(5,14)(6,23)(9,10)(11,13)(15,17).$$

Then, g_2 is in M_{23}, and we have $\sigma_r g_r = g_r \sigma_1$, $r = 1$, 2, $\sigma_2 g_1 = g_1 \sigma_2$ and $\sigma_1 g_2 = g_2 \sigma_2$.

Assume that $k = 9$, 10, 11, 15 or 16. From Table 11, we have that the degree of ρ_k is 8 or 24. Since σ_1 and σ_2 commute there exists a basis $\{v_l \,|\, 1 \le l \le \deg(\rho_k)\}$ of V_k, the vector space affording ρ_k, composed by simultaneous eigenvectors of $\rho_k(\sigma_1) = -\mathrm{Id}$ and $\rho_k(\sigma_2)$. Let us call $\rho_k(\sigma_2)v_l = \lambda_l v_l$, $1 \le l \le \deg(\rho_k)$, where $\lambda_l = \pm 1$, due to $|\sigma_2| = 2$. From Table 11, we have that $\sum_{l=1}^{\deg(\rho_k)} \lambda_l = 0$. Reordering the basis we can suppose that $\lambda_1 = \cdots = \lambda_{\deg(\rho_k)/2} = 1 = -\lambda_{1+\deg(\rho_k)/2} = \cdots = -\lambda_{\deg(\rho_k)}$. It is straightforward to check that if $W := \mathbb{C}$ - span of $\{g_1 v_l, g_2 v_l \,|\, 1 \le l \le \deg(\rho_k)\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_2}, \rho)$ of Cartan type whose associated Cartan matrix $\mathcal{A}$ has at least two row with three -1 or more. This means that the corresponding Dynkin diagram has at least two vertices with three edges or more; thus, $\mathcal{A}$ is not of finite type. Hence, $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \rho_k) = \infty$. $\square$

j	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert s_j\rvert$	1	2	2	3	3	4	4	4	5	6	6	7	7
$\lvert G^{s_j}\rvert$	$\lvert M_{24}\rvert$	21504	7680	1080	504	384	128	96	60	24	24	42	42
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2
χ_3	45	-3	5	0	3	-3	1	1	0	0	-1	A	A'
χ_4	45	-3	5	0	3	-3	1	1	0	0	-1	A'	A
χ_5	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0
χ_6	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0
χ_7	252	28	12	9	0	4	4	0	2	1	0	0	0
χ_8	253	13	-11	10	1	-3	1	1	3	-2	1	1	1
χ_9	483	35	3	6	0	3	3	3	-2	2	0	0	0
χ_{10}	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0
χ_{11}	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0
χ_{12}	990	-18	-10	0	3	6	2	-2	0	0	-1	A	A'
χ_{13}	990	-18	-10	0	3	6	2	-2	0	0	-1	A'	A
χ_{14}	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1
χ_{15}	1035	-21	-5	0	-3	3	3	-1	0	0	1	2 A	2 A'
χ_{16}	1035	-21	-5	0	-3	3	3	-1	0	0	1	2A'	2A
χ_{17}	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2
χ_{18}	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0
χ_{19}	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1
χ_{20}	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2
χ_{21}	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1
χ_{22}	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1
χ_{23}	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0
χ_{24}	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0
χ_{25}	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0
χ_{26}	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0

TABLE 12. Character table of M_{24} (i).

Remark 2.8. We compute that the groups $M_{23}^{s_3}$, $M_{23}^{s_4}$, $M_{23}^{s_5}$, $M_{23}^{s_7}$ and $M_{23}^{s_8}$ have 15, 14, 15, 14 and 14 conjugacy classes, respectively. Hence, there are 251 possible pairs $(\mathcal{O},\rho)$ for M_{23}; 248 of them lead to infinite-dimensional Nichols algebras, and 3 have negative braiding.

2.5. **The group M_{24}.** The Mathieu simple group M_{24} can be given as the subgroup of $\mathbb{S}_{24}$ generated by α_1, α_2 and α_3, where

$$\begin{aligned}
\alpha_1 &:= (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23),\\
\alpha_2 &:= (3,17,10,7,9)(4,13,14,19,5)(8,18,11,12,23)(15,20,22,21,16),\\
\alpha_3 &:= (1,24)(2,23)(3,12)(4,16)(5,18)(6,10)(7,20)(8,14)(9,21)(11,17)\\
&\qquad (13,22)(15,19).
\end{aligned}$$

The order of M_{24} is 244823040. In Tables 12 and 13, we show the character table of M_{24}, where $A = (-1+i\sqrt{7})/2$, $C = (-1+i\sqrt{15})/2$ and $D = (-1+i\sqrt{23})/2$. We will denote the representatives of the conjugacy classes of M_{24} by s_j, $1 \leq j \leq 26$.

In the following statement, we summarize our study by mean of abelian subracks in the group M_{24}.

j	14	15	16	17	18	19	20	21	22	23	24	25	26
$\lvert s_j\rvert$	8	10	11	12	12	14	14	15	15	21	21	23	23
$\lvert G^{s_j}\rvert$	16	20	11	12	12	14	14	15	15	21	21	23	23
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	-1	0	1	0	1	-A	-A'	0	0	A	A'	-1	-1
χ_4	-1	0	1	0	1	-A'	-A	0	0	A'	A	-1	-1
χ_5	-1	1	0	-1	0	0	0	C	C'	0	0	1	1
χ_6	-1	1	0	-1	0	0	0	C'	C	0	0	1	1
χ_7	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	0	0	0	-1	1	0	0	0	0	0	0	D	D'
χ_{11}	0	0	0	-1	1	0	0	0	0	0	0	D'	D
χ_{12}	0	0	0	0	1	A	A'	0	0	A	A'	1	1
χ_{13}	0	0	0	0	1	A'	A	0	0	A'	A	1	1
χ_{14}	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	-1	0	1	0	-1	0	0	0	0	-A	-A'	0	0
χ_{16}	-1	0	1	0	-1	0	0	0	0	-A'	-A	0	0
χ_{17}	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	1	0	0	0	0	0	0	0	0	0	0	-1	-1

TABLE 13. Character table of M_{24} (ii).

Theorem 2.9. *Let $\rho \in \widehat{M_{24}^{s_j}}$, with $1 \le j \le 26$. The braiding is negative in the cases $j = 6$, with $\rho = \rho_{2,6}$ or $\rho_{3,6}$, $j = 8$, with $\rho = \rho_{2,8}$ or $\rho_{3,8}$, $j = 14$, with $\rho = \epsilon \otimes \chi_{(-1)}$ or $\operatorname{sgn} \otimes \chi_{(-1)}$, $j = 17$, 18, 19 and 20, with $\rho = \chi_{(-1)}$. Otherwise, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$.*

Proof. *CASE:* $j = 4$, 5, 9, 16. From Tables 12 and 13, we see that s_j is real. By Lemma 1.3, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{24}^{s_j}}$.

CASE: $j = 12$, 13, 21, 22, 23, 24, 25, 26. We compute that s_j^2, $s_j^4 \in \mathcal{O}_{s_j}$. By Lemma 1.4, $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, for all $\rho \in \widehat{M_{24}^{s_j}}$, since $|s_j|$ is odd.

CASE: $j = 19$, 20. We have that $|s_j| = 14$ and $M_{24}^{s_j} \simeq \mathbb{Z}_{14}$. Although s_j is not real, we compute that s_j^9, $s_j^{11} \in \mathcal{O}_{s_j}$. Thus, if $q_{s_j s_j} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, by Lemma 1.4. The remained case corresponds to $\rho(s_j) = \omega_{14}^7 = -1$, which satisfies $q_{s_j s_j} = -1$. We compute that $\mathcal{O}_{s_j} \cap M_{24}^{s_j} = \{s_j, s_j^9, s_j^{11}\}$. It is straightforward to prove that the braiding is negative.

CASE: $j = 17,\ 18$. We have that $|s_j| = 12$ and $M_{24}^{s_j} \simeq \mathbb{Z}_{12}$. Also, we compute that s_j is real. Thus, if $q_{s_j s_j} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{s_j}, \rho) = \infty$, by Lemma 1.3. The remained case corresponds to $\rho(s_j) = \omega_{12}^6 = -1$, which satisfies $q_{s_j s_j} = -1$. We compute that $\mathcal{O}_{s_j} \cap M_{24}^{s_j} = \{s_j, s_j^5, s_j^7, s_j^{11}\}$. It is straightforward to prove that the braiding is negative.

CASE: $j = 15$. The representative s_{15} is

$$(1,11)(2,3,14,13,9,23,7,17,4,16)(5,19,10,20,18,6,21,15,22,8)(12,24),$$

it has order 10 and it is real. We compute that $M_{24}^{s_{15}} = \langle x, s_{15}\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_{10}$, where

$$x := (1,12)(2,18)(3,6)(4,10)(5,7)(8,23)(9,22)(11,24)(13,15)(14,21)$$
$$(16,20)(17,19).$$

Let us define $\{\nu_0, \dots, \nu_9\}$, where $\nu_l(s_{15}) := \omega_{10}^l$, $0 \le l \le 9$. So,

$$\widehat{M_{24}^{s_{15}}} = \{\epsilon \otimes \nu_l,\ \operatorname{sgn} \otimes \nu_l \mid 0 \le l \le 9\},$$

where ϵ and sgn mean the trivial and the sign representations of $\mathbb{Z}_2$, respectively. If $\rho \in \widehat{M_{24}^{s_{15}}}$, with $l \neq 5$, then $q_{s_{15}s_{15}} \neq -1$, and $\dim \mathfrak{B}(\mathcal{O}_{s_{15}}, \rho) = \infty$, by Lemma 1.3. The remained two cases are $\rho = \epsilon \otimes \nu_5$ and $\rho = \operatorname{sgn} \otimes \nu_5$. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_{15}}, \rho)$ is infinite-dimensional. First, we compute that $\mathcal{O}_{s_{15}} \cap M_{24}^{s_{15}}$ has 12 elements, and it contains $\sigma_1 := s_{15}$,

$$\sigma_2 := (1,12)(2,5,14,10,9,18,7,21,4,22)(3,19,13,20,23,6,17,15,16,8)(11,24),$$
$$\sigma_3 := (1,24)(2,6,14,15,9,8,7,19,4,20)(3,21,13,22,23,5,17,10,16,18)(11,12).$$

We compute that $\sigma_2\sigma_3 = \sigma_1^7$. We choose $g_1 = \mathrm{id}$,

$$g_2 := (1,11,24)(3,5,6)(8,23,18)(10,15,13)(16,22,20)(17,21,19)$$

and $g_3 := g_2^{-1}$. These elements are in M_{24} and they satisfy the same relations as in (2.6), (2.7) and (2.8).

Assume that $\rho = \epsilon \otimes \nu_5$ or $\operatorname{sgn} \otimes \nu_5$. If $W := \mathbb{C}$ - span of $\{g_1, g_2, g_3\}$, then W is a braided vector subspace of $M(\mathcal{O}_{s_{15}}, \rho)$ of Cartan type whose Cartan matrix is as in (2.5). Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_{15}}, \rho) = \infty$.

CASE: $j = 14$. The representative is

$$s_{14} = (1,22,18,14,11,19,16,7)(2,13)(3,12,6,21,8,10,20,23)(9,17,15,24),$$

it has order 8 and it is real. We compute that $M_{24}^{s_{14}} = \langle x, s_{14}\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_8$, where

$$x := (1,3)(2,13)(4,5)(6,18)(7,23)(8,11)(9,15)(10,19)(12,22)(14,21)$$
$$(16,20)(17,24).$$

Let us define $\{\nu_0,\dots,\nu_7\}$, where $\nu_l(s_{14}) := \omega_8^l$, $0\le l\le 7$. So,

$$\widehat{M_{24}^{s_{14}}} = \{\epsilon\otimes\nu_l,\ \operatorname{sgn}\otimes\nu_l \mid 0\le l\le 7\},$$

where ϵ and sgn mean the trivial and the sign representations of $\mathbb{Z}_2$, respectively. If $\rho\in\widehat{M_{24}^{s_{14}}}$, with $l\neq 4$, then $q_{s_{14}s_{14}}\neq -1$, and $\dim\mathfrak{B}(\mathcal{O}_{s_{14}},\rho)=\infty$, by Lemma 1.3. The remained two cases corresponding to $\rho=\epsilon\otimes\nu_4$ and $\rho=\operatorname{sgn}\otimes\nu_4$ have negative braiding. Indeed, we compute that

$$\mathcal{O}_{s_{14}}\cap M_{24}^{s_{14}} = \{s_{14}^{-1},\ xs_{14}^{-3},\ xs_{14},\ s_{14}^{3},\ s_{14}^{-3},\ xs_{14}^{3},\ s_{14},\ xs_{14}^{-1}\}.$$

For simplicity, we write $\sigma_1 := s_{14}^{-1}$, $\sigma_2 := xs_{14}^{-3}$, $\sigma_3 := xs_{14}$, $\sigma_4 := s_{14}^{3}$, $\sigma_5 := s_{14}^{-3}$, $\sigma_6 := xs_{14}^{3}$, $\sigma_7 := s_{14}$ and $\sigma_8 := xs_{14}^{-1}$. We choose in M_{24} the following elements:

$$\begin{aligned}
g_1 &:= (3,8)(4,5)(7,22)(9,15)(10,23)(12,21)(14,19)(16,18),\\
g_2 &:= (2,4)(5,13)(7,21)(9,17)(10,22)(12,19)(14,23)(15,24),\\
g_3 &:= (2,5)(4,13)(7,23)(9,24)(10,19)(12,22)(14,21)(15,17),\\
g_4 &:= (2,13)(3,8)(7,19)(10,21)(12,23)(14,22)(16,18)(17,24),\\
g_5 &:= (2,13)(4,5)(7,14)(9,15)(10,12)(17,24)(19,22)(21,23),\\
g_6 &:= (2,4,13,5)(3,8)(7,10,14,12)(9,24,15,17)(16,18)(19,23,22,21),
\end{aligned}$$

$g_7 := \mathrm{id}$ and $g_8 := g_6^{-1}$. We compute that these elements satisfy $\sigma_k g_7 = g_7\sigma_k$, $1\le k\le 8$, $\sigma_7 g_k = g_k\sigma_k$, $1\le k\le 5$, $\sigma_7 g_6 = g_6\sigma_8$ and $\sigma_7 g_8 = g_8\sigma_6$. It is easy to see that, if $\rho=\epsilon\otimes\nu_4$ or $\rho=\operatorname{sgn}\otimes\nu_4$, then $\rho(\gamma_{k,7}\gamma_{7,k})=1$, for every $1\le k\le 8$. From Lemma 1.8, the braiding is negative.

CASE: $j=10$. The representative s_{10} is

$$(1,20)(3,4,16)(5,14,21,19,23,15)(7,11,12,24,18,13)(8,22,10)(9,17),$$

it has order 6 and it is real. We compute that the centralizer $M_{24}^{s_{10}}$ is a non-abelian group of order 24.

Let $\rho=(\rho,V)\in\widehat{M_{24}^{s_{10}}}$. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_{10}},\rho)$ is infinite-dimensional. If $q_{s_{10}s_{10}}\neq -1$, then the result follows from Lemma 1.3. Assume that $q_{s_{10}s_{10}}=-1$. We compute that $\mathcal{O}_{s_{10}}\cap M_{24}^{s_{10}}$ has 10 elements and it contains $\sigma_1 := s_{10}$, $\sigma_2 := s_{10}^{-1}$,

$$\sigma_3 := (1,20)(2,6)(3,8,16,10,4,22)(5,14,21,19,23,15)(7,18,12)(11,13,24)$$

and $\sigma_4 := \sigma_3^{-1}$. These elements commute each other. We choose $g_1 := \mathrm{id}$, $g_2 := (2,6)(3,10)(4,22)(8,16)(11,13)(12,18)(14,15)(21,23)$,

$$g_3 := (1,6,17)(2,9,20)(3,7,5)(4,18,23)(8,11,14)(10,24,19)(12,21,16)(13,15,22)$$

and $g_4 := g_3g_2$. These elements are in M_{24} and they satisfy $\sigma_r g_r = g_r\sigma_1$, $\sigma_r g_1 = g_1\sigma_r$, $1 \le r \le 4$, and

$$\begin{aligned} &\sigma_1 g_2 = g_2\sigma_2, && \sigma_1 g_3 = g_3\gamma, && \sigma_1 g_4 = g_4\gamma^{-1},\\ &\sigma_3 g_2 = g_2\sigma_4, && \sigma_2 g_3 = g_3\gamma^{-1}, && \sigma_2 g_4 = g_4\gamma,\\ &\sigma_4 g_2 = g_2\sigma_3, && \sigma_4 g_3 = g_3\sigma_2, && \sigma_3 g_4 = g_4\sigma_2,\end{aligned}$$

where

$$\gamma := (2,6)(3,8,16,10,4,22)(5,23,21)(7,11,12,24,18,13)(9,17)(14,15,19).$$

Also, we compute that $\sigma_3\gamma = s_{10}^{-1}$. Now, we define $W := \mathbb{C}$ - span of $\{g_1v, g_2v, g_3v, g_4v\}$, where $v \in V - 0$. Hence, it is straightforward to check that W is a braided vector subspace of $M(\mathcal{O}_{s_{10}}, \rho)$ of Cartan type whose associated Cartan matrix is given by (2.11). By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_{10}}, \rho) = \infty$.

CASE: $j = 11$. We compute that $M_{24}^{s_{11}} \simeq M_{24}^{s_{10}}$. This implies that this case is analogous to the previous case, since $\mathcal{O}_{s_{11}} \simeq \mathcal{O}_{s_{10}}$ as racks.

CASE: $j = 8$. The representative s_8 is

$$(1,22,11,16)(2,8,14,17)(3,23,24,6)(4,21,12,13)(5,15,19,9)(7,18,20,10),$$

it has order 4 and it is real. We compute that the centralizer $M_{24}^{s_8}$ is a non-abelian group of order 96 whose character table is given by Table 14.

For every k, $1 \le k \le 20$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{24}^{s_8}$ whose character is μ_k. We compute that $s_8 \in \mathcal{O}_{20}^{M_{24}^{s_8}}$.

From Table 14, we have that if $k \neq 2$, 3, 10, 13, 14, then $q_{s_8s_8} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_8}, \rho_k) = \infty$, by Lemma 1.3. On the other hand, we compute that $\mathcal{O}_{s_8} \cap M_{24}^{s_8}$ has 32 elements and it contains

$$\begin{aligned}\sigma_1 &:= (1,2,11,14)(3,18,9,13)(4,23,20,5)(6,7,19,12)(8,16,17,22)(10,15,21,24),\\ \sigma_2 &:= (1,2,11,14)(3,21,9,10)(4,19,20,6)(5,7,23,12)(8,16,17,22)(13,15,18,24),\\ \sigma_5 &:= (1,16,11,22)(2,17,14,8)(3,5,24,19)(4,10,12,18)(6,9,23,15)(7,13,20,21),\end{aligned}$$

$\sigma_3 := \sigma_2^{-1}$, $\sigma_4 := \sigma_1^{-1}$, $\sigma_6 := s_8^{-1}$, $\sigma_7 := \sigma_5^{-1}$ and $\sigma_8 := s_8$. We choose in M_{24}

$$\begin{aligned} g_1 &:= (2,16,14,22)(3,9,15,24)(4,7)(5,21,19,10)(6,18,23,13)(8,17),\\ g_2 &:= (2,16,14,22)(4,12,7,20)(5,13,23,21)(6,10,19,18)(8,17)(9,24),\\ g_3 &:= (2,22,14,16)(3,9,15,24)(5,13,19,18)(6,10,23,21)(8,17)(12,20),\\ g_6 &:= (2,14)(3,4,15,7)(5,18,6,21)(9,20,24,12)(10,23,13,19)(16,22),\\ g_7 &:= (3,4)(5,13)(6,10)(7,15)(9,20)(12,24)(18,23)(19,21),\end{aligned}$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lvert y_k\rvert$	1	2	2	4	4	12	4	4	12	4	3	4	6	4	2	2	2	4	4	4
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	-1
μ_3	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1
μ_4	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	1	1	1	1
μ_5	1	-1	1	-i	i	i	-i	-i	-i	i	1	-1	-1	1	1	-1	-1	i	-i	i
μ_6	1	1	1	-i	-i	i	i	i	-i	-i	1	1	-1	-1	-1	-1	-1	i	-i	i
μ_7	1	-1	1	i	-i	-i	i	i	i	-i	1	-1	-1	1	1	-1	-1	-i	i	-i
μ_8	1	1	1	i	i	-i	-i	-i	i	i	1	1	-1	-1	-1	-1	-1	-i	i	-i
μ_9	2	0	2	2	0	-1	0	0	-1	0	-1	0	-1	0	0	2	2	2	2	2
μ_{10}	2	0	2	-2	0	1	0	0	1	0	-1	0	-1	0	0	2	2	-2	-2	-2
μ_{11}	2	0	2	-2i	0	-i	0	0	i	0	-1	0	1	0	0	-2	-2	2i	-2i	2i
μ_{12}	2	0	2	2i	0	i	0	0	-i	0	-1	0	1	0	0	-2	-2	-2i	2i	-2i
μ_{13}	3	-1	-1	1	-1	0	1	-1	0	1	0	1	0	1	-1	-1	3	1	-3	-3
μ_{14}	3	1	-1	1	1	0	-1	1	0	-1	0	-1	0	-1	1	-1	3	1	-3	-3
μ_{15}	3	-1	-1	-1	1	0	-1	1	0	-1	0	1	0	1	-1	-1	3	-1	3	3
μ_{16}	3	1	-1	-1	-1	0	1	-1	0	1	0	-1	0	-1	1	-1	3	-1	3	3
μ_{17}	3	-1	-1	i	-i	0	-i	i	0	i	0	1	0	-1	1	1	-3	-i	-3i	3i
μ_{18}	3	1	-1	i	i	0	i	-i	0	-i	0	-1	0	1	-1	1	-3	-i	-3i	3i
μ_{19}	3	-1	-1	-i	i	0	i	-i	0	-i	0	1	0	-1	1	1	-3	i	3i	-3i
μ_{20}	3	1	-1	-i	-i	0	-i	i	0	i	0	-1	0	1	-1	1	-3	i	3i	-3i

TABLE 14. Character table of $M_{24}^{s_8}$.

$g_4 := g_2^{-1}$, $g_5 := g_2^2$ and $g_8 :=$ id. We compute that these elements satisfy $\sigma_k g_l = g_l \gamma_{k,l}$, where $\gamma_{k,k} = s_8$, $\gamma_{k,8} = \sigma_k$, $1 \leq k \leq 8$, $\gamma_{8,1} = \sigma_3$, $\gamma_{8,2} = \sigma_4$, $\gamma_{8,3} = \sigma_1$, $\gamma_{8,4} = \sigma_2$, $\gamma_{8,5} = \sigma_5$, $\gamma_{8,6} = \sigma_6$, $\gamma_{8,7} = \sigma_7$, and

$$\begin{array}{llllll}
\gamma_{1,2} = \sigma_7, & \gamma_{1,3} = \sigma_5, & \gamma_{1,4} = \sigma_6, & \gamma_{1,5} = \sigma_3, & \gamma_{1,6} = \sigma_3, & \gamma_{1,7} = \sigma_1, \\
\gamma_{2,1} = \sigma_7, & \gamma_{2,3} = \sigma_6, & \gamma_{2,4} = \sigma_5, & \gamma_{2,5} = \sigma_4, & \gamma_{2,6} = \sigma_4, & \gamma_{2,7} = \sigma_2, \\
\gamma_{3,1} = \sigma_5, & \gamma_{3,2} = \sigma_6, & \gamma_{3,4} = \sigma_7, & \gamma_{3,5} = \sigma_1, & \gamma_{3,6} = \sigma_1, & \gamma_{3,7} = \sigma_3, \\
\gamma_{4,1} = \sigma_6, & \gamma_{4,2} = \sigma_5, & \gamma_{4,3} = \sigma_7, & \gamma_{4,5} = \sigma_2, & \gamma_{4,6} = \sigma_2, & \gamma_{4,7} = \sigma_4, \\
\gamma_{5,1} = \sigma_1, & \gamma_{5,2} = \sigma_2, & \gamma_{5,3} = \sigma_3, & \gamma_{5,4} = \sigma_4, & \gamma_{5,6} = \sigma_7, & \gamma_{5,7} = \sigma_6, \\
\gamma_{6,1} = \sigma_2, & \gamma_{6,2} = \sigma_1, & \gamma_{6,3} = \sigma_4, & \gamma_{6,4} = \sigma_3, & \gamma_{6,5} = \sigma_7, & \gamma_{6,7} = \sigma_5, \\
\gamma_{7,1} = \sigma_4, & \gamma_{7,2} = \sigma_3, & \gamma_{7,3} = \sigma_2, & \gamma_{7,4} = \sigma_1, & \gamma_{7,5} = \sigma_6, & \gamma_{7,6} = \sigma_5.
\end{array}$$

Assume that $k = 10$, 13 or 14; thus, $q_{s_8 s_8} = -1$. We check *case by case* that always we can construct a braided vector subspace of $M(\mathcal{O}_{s_8}, \rho_k)$ of diagonal type whose generalized Dynkin diagram contains an r-cycle with $r > 3$ or a vertex with valency greater than 3. By Lemma 1.5, $\dim \mathfrak{B}(\mathcal{O}_{s_8}, \rho_k) = \infty$.

Finally, assume that $k = 2$ or 3. Thus, $q_{s_8 s_8} = -1$ and we compute that $\rho(\gamma_{1,t}\gamma_{t,1}) = 1$, for every $1 \leq t \leq 32$. By Lemma 1.8, the braiding is negative.

CASE: $j = 7$. The representative is

$$s_7 = (1, 18, 11, 16)(3, 6, 8, 20)(7, 22, 14, 19)(9, 15)(10, 23, 12, 21)(17, 24),$$

it has order 4 and it is real. We compute that the centralizer $M_{24}^{s_7}$ is a non-abelian group of order 128.

Let $\rho = (\rho, V) \in \widehat{M_{24}^{s_7}}$. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_7}, \rho)$ is infinite-dimensional. If $q_{s_7 s_7} \neq -1$, then the result follows from Lemma 1.3. Assume that $q_{s_7 s_7} = -1$. We compute that $\mathcal{O}_{s_7} \cap M_{24}^{s_7}$ has 40 elements and it contains

$$\sigma_1 := (2, 4, 13, 5)(3, 8)(6, 20)(7, 19, 14, 22)(9, 17, 15, 24)(10, 23, 12, 21),$$
$$\sigma_3 := (1, 16, 11, 18)(2, 5, 13, 4)(3, 6, 8, 20)(9, 17, 15, 24)(10, 12)(21, 23),$$

$\sigma_2 := \sigma_1^{-1}$ and $\sigma_4 := \sigma_3^{-1}$. These elements commute each other and $\sigma_1\sigma_3 = s_7^{-1}$. Now, we choose

$$g_1 := (1, 2, 18, 4, 11, 13, 16, 5)(3, 9, 20, 24, 8, 15, 6, 17)(10, 21, 12, 23)(19, 22),$$
$$g_2 := (1, 2, 11, 13)(3, 9, 8, 15)(4, 18, 5, 16)(6, 24, 20, 17)(10, 23)(12, 21),$$
$$g_3 := (2, 19, 4, 14, 13, 22, 5, 7)(3, 6, 8, 20)(9, 23, 17, 12, 15, 21, 24, 10)(16, 18),$$
$$g_4 := (2, 14, 13, 7)(3, 6)(4, 19, 5, 22)(8, 20)(9, 10, 15, 12)(17, 21, 24, 23).$$

These elements are in M_{24} and they satisfy

$$\begin{array}{llll} \sigma_1 g_1 = g_1 s_7, & \sigma_1 g_2 = g_2 s_7^{-1}, & \sigma_1 g_3 = g_3 \sigma_1, & \sigma_1 g_4 = g_4 \sigma_2, \\ \sigma_2 g_1 = g_1 s_7^{-1}, & \sigma_2 g_2 = g_2 s_7, & \sigma_2 g_3 = g_3 \sigma_2, & \sigma_2 g_4 = g_4 \sigma_1, \\ \sigma_3 g_1 = g_1 \sigma_3, & \sigma_3 g_2 = g_2 \sigma_4, & \sigma_3 g_3 = g_3 s_7, & \sigma_3 g_4 = g_4 s_7^{-1}, \\ \sigma_4 g_1 = g_1 \sigma_4, & \sigma_4 g_2 = g_2 \sigma_3, & \sigma_4 g_3 = g_3 s_7^{-1}, & \sigma_4 g_4 = g_4 s_7. \end{array}$$

We define $W := \mathbb{C}$ - span of $\{g_1 v, g_2 v, g_3 v, g_4 v\}$, where $v \in V - 0$. Hence, it is straightforward to check that W is a braided vector subspace of $M(\mathcal{O}_{s_7}, \rho)$ of Cartan type whose associated Cartan matrix is given by (2.11). By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{s_7}, \rho) = \infty$.

CASE: $j = 6$. The representative s_6 is

$$(1, 9, 20, 17)(2, 6)(3, 10)(4, 8)(5, 24, 19, 7)(11, 14, 18, 23)(12, 21, 13, 15)(16, 22)$$

and it is real. We compute that $M_{24}^{s_6}$ is a non-abelian group of order 384 whose character table is given by Tables 15 and 16.

For every k, $1 \leq k \leq 26$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{24}^{s_6}$ whose character is μ_k. We compute that $s_6 \in \mathcal{O}_{23}^{M_{24}^{s_6}}$. From Tables 15 and 16, we have that if $k \neq 2$, 3, 5, 9, 10, 13, 14, 15, 16, 24, then $q_{s_6 s_6} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho_k) = \infty$, by Lemma 1.3.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert y_k \rvert$	1	3	2	2	4	2	2	6	4	4	4	2	4
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	1
μ_3	1	1	1	1	1	1	1	1	1	1	1	1	-1
μ_4	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1
μ_5	2	-1	2	0	0	2	2	-1	2	0	0	0	0
μ_6	2	-1	2	0	0	2	2	-1	2	0	0	0	0
μ_7	3	0	-1	-1	1	3	-1	0	-1	-1	1	-1	-1
μ_8	3	0	-1	1	-1	3	-1	0	-1	1	-1	1	1
μ_9	3	0	-1	-1	1	3	-1	0	-1	-1	1	-1	1
μ_{10}	3	0	-1	1	-1	3	-1	0	-1	1	-1	1	-1
μ_{11}	3	0	3	1	1	-1	-1	0	-1	-1	-1	1	1
μ_{12}	3	0	-1	1	-1	-1	3	0	-1	-1	1	1	1
μ_{13}	3	0	-1	-1	1	-1	3	0	-1	1	-1	-1	1
μ_{14}	3	0	-1	1	-1	-1	3	0	-1	-1	1	1	-1
μ_{15}	3	0	3	-1	-1	-1	-1	0	-1	1	1	-1	1
μ_{16}	3	0	3	1	1	-1	-1	0	-1	-1	-1	1	-1
μ_{17}	3	0	-1	-1	1	-1	3	0	-1	1	-1	-1	-1
μ_{18}	3	0	3	-1	-1	-1	-1	0	-1	1	1	-1	-1
μ_{19}	4	1	0	2	0	0	0	-1	0	0	0	-2	2i
μ_{20}	4	1	0	-2	0	0	0	-1	0	0	0	2	-2i
μ_{21}	4	1	0	-2	0	0	0	-1	0	0	0	2	2i
μ_{22}	4	1	0	2	0	0	0	-1	0	0	0	-2	-2i
μ_{23}	6	0	-2	0	0	-2	-2	0	2	0	0	0	0
μ_{24}	6	0	-2	0	0	-2	-2	0	2	0	0	0	0
μ_{25}	8	-1	0	0	0	0	0	1	0	0	0	0	0
μ_{26}	8	-1	0	0	0	0	0	1	0	0	0	0	0

TABLE 15. Character table of $M_{24}^{s_6}$ (i).

We compute that $\mathcal{O}_{s_6} \cap M_{24}^{s_6}$ has 80 elements and it contains

$$\sigma_1 := (1,7)(2,4,3,16)(5,9)(6,8,10,22)(11,14,18,23)(12,15,13,21)(17,19)(20,24),$$
$$\sigma_4 := (1,9,20,17)(2,10)(3,6)(4,22)(5,24,19,7)(8,16)(11,23,18,14)(12,15,13,21),$$
$$\sigma_7 := (1,24)(2,4,3,16)(5,17)(6,8,10,22)(7,20)(9,19)(11,23,18,14)(12,21,13,15),$$

$\sigma_2 := \sigma_1^{-1}$, $\sigma_3 := s_6$, $\sigma_5 := s_6^{-1}$ $\sigma_6 := \sigma_4^{-1}$ and $\sigma_8 := \sigma_7^{-1}$. We compute that σ_1, σ_2, σ_7, $\sigma_8 \in \mathcal{O}_{17}^{M_{24}^{s_6}}$, σ_4, $\sigma_6 \in \mathcal{O}_{24}^{M_{24}^{s_6}}$ and $s_6^{-1} \in \mathcal{O}_{25}^{M_{24}^{s_6}}$. We choose in M_{24}

$$g_1 := (1,2,17,16)(3,9,4,20)(5,8,24,10)(6,19,22,7)(11,12,23,21)(13,14,15,18),$$
$$g_4 := (2,4)(3,16)(6,22)(8,10)(11,14)(12,21)(13,15)(18,23),$$
$$g_5 := (2,10)(3,6)(4,22)(7,24)(8,16)(9,17)(14,23)(15,21),$$
$$g_7 := (1,2,24,10,20,3,7,6)(4,5,8,17,16,19,22,9)(12,15,13,21)(14,23),$$

$g_3 :=$ id, $g_2 := g_1 g_5$, $g_6 := g_4 g_5$ and $g_8 := g_7 g_5$. We compute that these elements satisfy $\sigma_k g_l = g_l \gamma_{k,l}$, where $\gamma_{k,k} = s_6$, $\gamma_{k,3} = \sigma_k$, $1 \leq k \leq 8$, $\gamma_{3,1} = \sigma_2$, $\gamma_{3,2} = \sigma_8$, $\gamma_{3,4} = \sigma_4$, $\gamma_{3,5} = \sigma_5$, $\gamma_{3,6} = \sigma_6$, $\gamma_{3,7} = \sigma_2$, $\gamma_{3,8} = \sigma_8$,

k	14	15	16	17	18	19	20	21	22	23	24	25	26
$\lvert y_k\rvert$	4	4	4	4	12	4	12	2	4	4	4	4	2
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1
μ_3	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1
μ_4	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1
μ_5	0	0	0	0	1	-2	1	-2	-2	-2	-2	-2	2
μ_6	0	0	0	0	-1	2	-1	2	2	2	2	2	2
μ_7	-1	-1	1	1	0	3	0	-1	-1	3	-1	3	3
μ_8	1	1	-1	-1	0	3	0	-1	-1	3	-1	3	3
μ_9	1	1	-1	-1	0	-3	0	1	1	-3	1	-3	3
μ_{10}	-1	-1	1	1	0	-3	0	1	1	-3	1	-3	3
μ_{11}	-1	1	-1	1	0	-1	0	-1	-1	3	3	3	3
μ_{12}	-1	1	1	-1	0	-1	0	-1	3	3	-1	3	3
μ_{13}	-1	1	1	-1	0	1	0	1	-3	-3	1	-3	3
μ_{14}	1	-1	-1	1	0	1	0	1	-3	-3	1	-3	3
μ_{15}	-1	1	-1	1	0	1	0	1	1	-3	-3	-3	3
μ_{16}	1	-1	1	-1	0	1	0	1	1	-3	-3	-3	3
μ_{17}	1	-1	-1	1	0	-1	0	-1	3	3	-1	3	3
μ_{18}	1	-1	1	-1	0	-1	0	-1	-1	3	3	3	3
μ_{19}	0	-2i	0	0	i	0	-i	0	0	-4i	0	4i	-4
μ_{20}	0	2i	0	0	i	0	-i	0	0	-4i	0	4i	-4
μ_{21}	0	-2i	0	0	-i	0	i	0	0	4i	0	-4i	-4
μ_{22}	0	2i	0	0	-i	0	i	0	0	4i	0	-4i	-4
μ_{23}	0	0	0	0	0	-2	0	2	-2	6	-2	6	6
μ_{24}	0	0	0	0	0	2	0	-2	2	-6	2	-6	6
μ_{25}	0	0	0	0	-i	0	i	0	0	-8i	0	8i	-8
μ_{26}	0	0	0	0	i	0	-i	0	0	8i	0	-8i	-8

TABLE 16. Character table of $M_{24}^{s_6}$ (ii).

and

$$
\begin{array}{llllll}
\gamma_{1,2}=\sigma_5, & \gamma_{1,4}=\sigma_2, & \gamma_{1,5}=\sigma_7, & \gamma_{1,6}=\sigma_8, & \gamma_{1,7}=\sigma_4, & \gamma_{1,8}=\sigma_6,\\
\gamma_{2,1}=\sigma_5, & \gamma_{2,4}=\sigma_1, & \gamma_{2,5}=\sigma_8, & \gamma_{2,6}=\sigma_7, & \gamma_{2,7}=\sigma_6, & \gamma_{2,8}=\sigma_4,\\
\gamma_{4,1}=\sigma_8, & \gamma_{4,2}=\sigma_2, & \gamma_{4,5}=\sigma_6, & \gamma_{4,6}=\sigma_5, & \gamma_{4,7}=\sigma_8, & \gamma_{4,8}=\sigma_2,\\
\gamma_{5,1}=\sigma_1, & \gamma_{5,2}=\sigma_7, & \gamma_{5,4}=\sigma_6, & \gamma_{5,6}=\sigma_4, & \gamma_{5,7}=\sigma_1, & \gamma_{5,8}=\sigma_7,\\
\gamma_{6,1}=\sigma_7, & \gamma_{6,2}=\sigma_1, & \gamma_{6,4}=\sigma_5, & \gamma_{6,5}=\sigma_4, & \gamma_{6,7}=\sigma_7, & \gamma_{6,8}=\sigma_1,\\
\gamma_{7,1}=\sigma_4, & \gamma_{7,2}=\sigma_6, & \gamma_{7,4}=\sigma_8, & \gamma_{7,5}=\sigma_1, & \gamma_{7,6}=\sigma_2, & \gamma_{7,5}=\sigma_5,\\
\gamma_{8,1}=\sigma_6, & \gamma_{8,2}=\sigma_4, & \gamma_{8,4}=\sigma_7, & \gamma_{8,5}=\sigma_2, & \gamma_{8,6}=\sigma_1, & \gamma_{8,7}=\sigma_5.
\end{array}
$$

Assume that $k = 5$, 9, 10, 13, 14, 15, 16 or 24; thus, $q_{s_6s_6} = -1$. We check *case by case* that always we can construct a braided vector subspace of $M(\mathcal{O}_{s_6}, \rho_k)$ of diagonal type whose generalized Dynkin diagram contains an r-cycle with $r > 3$ or a vertex with valency greater than 3. By Lemma 1.5, $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho_k) = \infty$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\|y_k\|$	1	2	3	4	7	7	6	2	2	4	4	2	4	4	6
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	3	-1	0	1	A	A’	0	-1	3	1	-1	3	-1	-1	0
μ_3	3	-1	0	1	A’	A	0	-1	3	1	-1	3	-1	-1	0
μ_4	6	2	0	0	-1	-1	0	2	6	0	2	6	2	2	0
μ_5	7	-1	1	-1	0	0	-1	3	-1	1	-1	7	-1	-1	1
μ_6	7	3	1	1	0	0	1	3	7	1	3	-1	-1	-1	1
μ_7	7	-1	1	-1	0	0	1	-1	7	-1	-1	7	-1	-1	1
μ_8	7	3	1	1	0	0	-1	-1	-1	-1	-1	7	3	3	1
μ_9	7	-1	1	-1	0	0	1	-1	7	-1	-1	-1	-1	3	1
μ_{10}	8	0	-1	0	1	1	-1	0	8	0	0	8	0	0	-1
μ_{11}	8	0	2	0	1	1	0	4	0	2	0	0	0	0	-2
μ_{12}	14	2	-1	0	0	0	1	2	-2	0	-2	14	2	2	-1
μ_{13}	14	2	-1	0	0	0	-1	2	14	0	2	-2	-2	2	-1
μ_{14}	21	1	0	-1	0	0	0	5	-3	1	-3	-3	1	-3	0
μ_{15}	21	-3	0	1	0	0	0	1	-3	-1	1	21	-3	-3	0
μ_{16}	21	1	0	-1	0	0	0	1	21	-1	1	-3	1	-3	0
μ_{17}	21	1	0	-1	0	0	0	-3	-3	1	1	-3	-3	5	0
μ_{18}	21	-3	0	1	0	0	0	1	-3	-1	1	-3	1	1	0
μ_{19}	21	5	0	1	0	0	0	1	-3	-1	-3	-3	-3	1	0
μ_{20}	21	1	0	-1	0	0	0	-3	-3	1	1	21	1	1	0
μ_{21}	21	-3	0	1	0	0	0	-3	21	1	-3	-3	1	1	0
μ_{22}	24	0	0	0	A’	A	0	-4	0	2	0	0	0	0	0
μ_{23}	24	0	0	0	A	A’	0	-4	0	2	0	0	0	0	0
μ_{24}	28	-4	1	0	0	0	-1	4	-4	0	0	-4	0	4	1
μ_{25}	28	4	1	0	0	0	-1	-4	-4	0	0	-4	0	-4	1
μ_{26}	42	-2	0	0	0	0	0	-2	-6	0	2	-6	2	-2	0
μ_{27}	48	0	0	0	-1	-1	0	8	0	0	0	0	0	0	0
μ_{28}	56	0	-1	0	0	0	1	0	-8	0	0	-8	0	0	-1
μ_{29}	56	0	2	0	0	0	0	-4	0	-2	0	0	0	0	-2
μ_{30}	64	0	-2	0	1	1	0	0	0	0	0	0	0	0	2

TABLE 17. Character table of $M_{24}^{s_2}$ (i).

Finally, assume that $k = 2$ or 3. Thus, $q_{s_6 s_6} = -1$ and we compute that $\rho(\gamma_{1,t}\gamma_{t,1}) = 1$, for every $1 \leq t \leq 80$. By Lemma 1.8, the braiding is negative.

CASE: $j = 2$. The representative is

$$s_2 = (1, 20)(5, 19)(7, 24)(9, 17)(11, 18)(12, 13)(14, 23)(15, 21).$$

We compute that the centralizer $M_{24}^{s_2}$ is a non-abelian group of order 21504 whose character table is given by Tables 17 and 18, where $A = (-1+i\sqrt{7})/2$.

For every k, $1 \leq k \leq 30$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{24}^{s_2}$ whose character is μ_k. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_2}, \rho_k)$ is infinite-dimensional, for every k, $1 \leq k \leq 30$. We compute that $s_2 \in \mathcal{O}_{30}^{M_{24}^{s_2}}$. If $k \neq 11$, 22, 23, 27, 29, 30, then the result follows from Lemma 1.3, because $q_{s_2 s_2} \neq -1$. Assume that $k = 11$, 22, 23, 27, 29 or 30; thus, $q_{s_2 s_2} = -1$. From Tables 17 and 18, we have that $8 \leq \deg(\rho_k)$ even.

k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\lvert y_k\rvert$	8	14	6	14	4	2	2	4	12	4	4	4	2	4	2
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	A	0	A'	-1	3	-1	1	0	1	-1	-1	-1	3	3
μ_3	1	A'	0	A	-1	3	-1	1	0	1	-1	-1	-1	3	3
μ_4	0	-1	0	-1	2	6	2	0	0	0	2	2	2	6	6
μ_5	-1	0	1	0	3	-1	3	1	-1	1	-1	-1	3	-1	7
μ_6	-1	0	-1	0	-1	-1	3	-1	-1	1	-1	-1	-1	-1	7
μ_7	-1	0	1	0	-1	7	-1	-1	1	-1	-1	-1	-1	7	7
μ_8	1	0	1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	7
μ_9	1	0	-1	0	-1	-1	-1	1	-1	-1	-1	3	3	-1	7
μ_{10}	0	1	-1	1	0	8	0	0	-1	0	0	0	0	8	8
μ_{11}	0	-1	0	-1	0	0	-4	0	0	-2	0	0	0	0	-8
μ_{12}	0	0	-1	0	2	-2	2	0	1	0	-2	-2	2	-2	14
μ_{13}	0	0	1	0	-2	-2	2	0	1	0	-2	2	2	-2	14
μ_{14}	1	0	0	0	-3	5	5	-1	0	1	1	1	1	-3	21
μ_{15}	1	0	0	0	1	-3	1	-1	0	-1	1	1	1	-3	21
μ_{16}	1	0	0	0	1	-3	1	1	0	-1	1	-3	-3	-3	21
μ_{17}	1	0	0	0	1	5	-3	-1	0	1	1	-3	1	-3	21
μ_{18}	-1	0	0	0	-3	5	1	1	0	-1	1	-3	5	-3	21
μ_{19}	-1	0	0	0	1	5	1	1	0	-1	1	1	-3	-3	21
μ_{20}	-1	0	0	0	-3	-3	-3	1	0	1	1	1	-3	-3	21
μ_{21}	-1	0	0	0	1	-3	-3	-1	0	1	1	1	1	-3	21
μ_{22}	0	-A'	0	-A	0	0	4	0	0	-2	0	0	0	0	-24
μ_{23}	0	-A	0	-A'	0	0	4	0	0	-2	0	0	0	0	-24
μ_{24}	0	0	-1	0	0	-4	4	0	1	0	0	0	-4	4	28
μ_{25}	0	0	-1	0	0	-4	-4	0	1	0	0	0	4	4	28
μ_{26}	0	0	0	0	2	10	-2	0	0	0	-2	2	-2	-6	42
μ_{27}	0	1	0	1	0	0	-8	0	0	0	0	0	0	0	-48
μ_{28}	0	0	1	0	0	-8	0	0	-1	0	0	0	0	8	56
μ_{29}	0	0	0	0	0	0	4	0	0	2	0	0	0	0	-56
μ_{30}	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	-64

TABLE 18. Character table of $M_{24}^{s_2}$ (ii).

We compute that $\mathcal{O}_{s_2} \cap M_{24}^{s_2}$ has 281 elements and it contains $\sigma_1 := s_2$ and

$$\sigma_2 := (2,16)(3,4)(5,19)(6,22)(7,24)(8,10)(11,18)(14,23),$$

with $\sigma_2 \in \mathcal{O}_9^{M_{24}^{s_2}}$. We choose $g_1 := \mathrm{id}$ and

$$g_2 := (1,2)(3,13,22,17,10,15)(4,12,6,9,8,21)(5,11,7)(16,20)(18,24,19).$$

These elements are in M_{24} and they satisfy $\sigma_1 g_1 = g_1 \sigma_1$, $\sigma_2 g_1 = g_1 \sigma_2$, $\sigma_1 g_2 = g_2 \sigma_2$ and $\sigma_2 g_2 = g_2 \sigma_1$. Since σ_1 and σ_2 commute there exists a basis $\{v_l \,|\, 1 \leq l \leq \deg(\rho_k)\}$ of V_k, the vector space affording ρ_k, composed by simultaneous eigenvectors of $\rho_k(\sigma_1) = -\mathrm{Id}$ and $\rho_k(\sigma_2)$. Let us call $\rho_k(\sigma_2) v_l = \lambda_l v_l$, $1 \leq l \leq \deg(\rho_k)$, where $\lambda_l = \pm 1$, due to $|\sigma_2| = 2$. From Table 17, we have that $\sum_{l=1}^{\deg(\rho_k)} \lambda_l = 0$. Reordering the basis we can suppose that $\lambda_1 = \cdots = \lambda_{\deg(\rho_k)/2} = 1 = -\lambda_{1+\deg(\rho_k)/2} = \cdots = -\lambda_{\deg(\rho_k)}$. It is straightforward to check that if $W := \mathbb{C}$ - span of $\{g_1 v_l, g_2 v_l \,|\, 1 \leq l \leq \deg(\rho_k)\}$, then W

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert y_k\rvert$	1	5	10	10	10	3	6	12	6	6	2	4	8	8	4
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	1	1	1	1	1	1	1	-1	-1	1	1	1	-1	-1	1
μ_3	4	-1	-1	-1	-1	1	1	1	1	1	4	0	0	0	0
μ_4	4	-1	-1	-1	-1	1	1	-1	-1	1	4	0	0	0	0
μ_5	5	0	0	0	0	-1	-1	1	1	-1	5	1	-1	-1	1
μ_6	5	0	0	0	0	-1	-1	-1	-1	-1	5	1	1	1	1
μ_7	6	1	1	1	1	0	0	0	0	0	6	-2	0	0	-2
μ_8	6	1	1	-1	-1	0	0	0	0	0	-2	2	0	0	-2
μ_9	6	1	1	-1	-1	0	0	0	0	0	-2	2	0	0	-2
μ_{10}	6	1	1	-1	-1	0	0	0	0	0	-2	-2	2i	-2i	2
μ_{11}	6	1	1	-1	-1	0	0	0	0	0	-2	-2	-2i	2i	2
μ_{12}	10	0	0	0	0	1	1	-1	1	-1	2	2	0	0	-2
μ_{13}	10	0	0	0	0	1	1	1	-1	-1	2	2	0	0	-2
μ_{14}	10	0	0	0	0	1	1	-1	1	-1	2	-2	0	0	2
μ_{15}	10	0	0	0	0	1	1	1	-1	-1	2	-2	0	0	2
μ_{16}	12	2	-2	0	0	0	0	0	0	0	4	0	0	0	0
μ_{17}	12	2	-2	0	0	0	0	0	0	0	4	0	0	0	0
μ_{18}	15	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1
μ_{19}	15	0	0	0	0	0	0	0	0	0	-1	3	1	1	3
μ_{20}	15	0	0	0	0	0	0	0	0	0	-1	3	-1	-1	3
μ_{21}	15	0	0	0	0	0	0	0	0	0	-1	-1	1	1	-1
μ_{22}	20	0	0	0	0	-1	-1	-1	1	1	4	0	0	0	0
μ_{23}	20	0	0	0	0	-1	-1	1	-1	1	4	0	0	0	0
μ_{24}	20	0	0	0	0	2	-2	0	0	0	-4	0	0	0	0
μ_{25}	20	0	0	0	0	2	-2	0	0	0	-4	0	0	0	0
μ_{26}	24	-1	-1	1	1	0	0	0	0	0	-8	0	0	0	0
μ_{27}	24	-1	1	A	-A	0	0	0	0	0	8	0	0	0	0
μ_{28}	24	-1	1	-A	A	0	0	0	0	0	8	0	0	0	0
μ_{29}	30	0	0	0	0	0	0	0	0	0	-2	-2	0	0	-2
μ_{30}	40	0	0	0	0	-2	2	0	0	0	-8	0	0	0	0

TABLE 19. Character table of $M_{24}^{s_3}$ (i).

is a braided vector subspace of $M(\mathcal{O}_{s_2}, \rho)$ of Cartan type whose associated Cartan matrix $\mathcal{A}$ has at least two row with three -1 or more. This means that the corresponding Dynkin diagram has at least two vertices with three edges or more; thus, $\mathcal{A}$ is not of finite type. Hence, $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \rho_k) = \infty$.

CASE: $j = 3$. The representative s_3 is

$$(1, 11)(2, 23)(3, 7)(4, 13)(5, 6)(8, 18)(9, 16)(10, 15)(12, 24)(14, 17)(19, 21)(20, 22).$$

We compute that the centralizer $M_{24}^{s_3}$ is a non-abelian group of order 7680 whose character table is given by Tables 19 and 20, where $A = -i\sqrt{5}$.

For every k, $1 \leq k \leq 30$, we call $\rho_k = (\rho_k, V_k)$ the irreducible representation of $M_{24}^{s_3}$ whose character is μ_k. We will prove that the Nichols algebra $\mathfrak{B}(\mathcal{O}_{s_3}, \rho_k)$ is infinite-dimensional, for every k, $1 \leq k \leq 30$. We compute that $s_3 \in \mathcal{O}_{17}^{M_{24}^{s_3}}$. Now, if $k \neq 16$, 17, 24, 25, 27, 28, 30, then the result

k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\lvert y_k \rvert$	4	2	4	2	4	2	4	4	2	2	4	2	4	2	2
μ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
μ_2	-1	1	-1	1	-1	1	1	1	1	1	-1	1	-1	1	-1
μ_3	-2	4	-2	4	-2	4	0	0	4	0	0	0	0	0	-2
μ_4	2	4	2	4	2	4	0	0	4	0	0	0	0	0	2
μ_5	1	5	1	5	1	5	1	1	5	1	-1	1	-1	1	1
μ_6	-1	5	-1	5	-1	5	1	1	5	1	1	1	1	1	-1
μ_7	0	6	0	6	0	6	-2	-2	6	-2	0	-2	0	-2	0
μ_8	0	6	0	-2	0	2	0	0	-6	2	2	2	-2	-2	0
μ_9	0	6	0	-2	0	2	0	0	-6	2	-2	2	2	-2	0
μ_{10}	0	6	0	-2	0	2	0	0	-6	-2	0	-2	0	2	0
μ_{11}	0	6	0	-2	0	2	0	0	-6	-2	0	-2	0	2	0
μ_{12}	2	10	2	2	-2	-2	0	0	-10	-2	0	-2	0	2	-2
μ_{13}	-2	10	-2	2	2	-2	0	0	-10	-2	0	-2	0	2	2
μ_{14}	0	10	-4	2	0	-2	0	0	-10	2	0	2	0	-2	4
μ_{15}	0	10	4	2	0	-2	0	0	-10	2	0	2	0	-2	-4
μ_{16}	0	-12	0	-4	0	0	2	-2	0	4	0	-4	0	0	0
μ_{17}	0	-12	0	-4	0	0	-2	2	0	-4	0	4	0	0	0
μ_{18}	-1	15	3	-1	-1	-1	-1	-1	15	3	1	3	1	3	3
μ_{19}	-1	15	3	-1	-1	-1	-1	-1	15	-1	-1	-1	-1	-1	3
μ_{20}	1	15	-3	-1	1	-1	-1	-1	15	-1	1	-1	1	-1	-3
μ_{21}	1	15	-3	-1	1	-1	-1	-1	15	3	-1	3	-1	3	-3
μ_{22}	-2	20	2	4	2	-4	0	0	-20	0	0	0	0	0	-2
μ_{23}	2	20	-2	4	-2	-4	0	0	-20	0	0	0	0	0	2
μ_{24}	0	-20	0	4	0	0	-2	2	0	4	0	-4	0	0	0
μ_{25}	0	-20	0	4	0	0	2	-2	0	-4	0	4	0	0	0
μ_{26}	0	24	0	-8	0	8	0	0	-24	0	0	0	0	0	0
μ_{27}	0	-24	0	-8	0	0	0	0	0	0	0	0	0	0	0
μ_{28}	0	-24	0	-8	0	0	0	0	0	0	0	0	0	0	0
μ_{29}	0	30	0	-2	0	-2	2	2	30	-2	0	-2	0	-2	0
μ_{30}	0	-40	0	8	0	0	0	0	0	0	0	0	0	0	0

TABLE 20. Character table of $M_{24}^{s_3}$ (ii).

follows from Lemma 1.3, because $q_{s_3s_3} \neq -1$. Assume that $k = 16$, 17, 24, 25, 27, 28 or 30; thus, $q_{s_3s_3} = -1$. We compute that $\mathcal{O}_{s_3} \cap M_{24}^{s_3}$ has 278 elements and it contains $\sigma_1 := s_3$ and

$$\begin{aligned}\sigma_2 := {} & (1,5)(2,10)(3,12)(4,8)(6,11)(7,24)(9,19)(13,18)(14,22)(15,23)\\ & (16,21)(17,20),\end{aligned}$$

with $\sigma_2 \in \mathcal{O}_{30}^{M_{24}^{s_3}}$. We choose $g_1 := \mathrm{id}$ and

$$g_2 := (5,11)(7,12)(8,13)(9,20)(10,23)(14,21)(16,17)(19,22).$$

These elements are in M_{24} and they satisfy $\sigma_1 g_1 = g_1\sigma_1$, $\sigma_2 g_1 = g_1\sigma_2$, $\sigma_1 g_2 = g_2\sigma_2$ and $\sigma_2 g_2 = g_2\sigma_1$. Since $12 \leq \deg(\rho_k)$ even and $\mu_k(\mathcal{O}_{30}^{M_{24}^{s_3}}) = 0$, for $k = 16$, 17, 24, 25, 27, 28 or 30, we can proceed as in the previous case. Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_3}, \rho_k) = \infty$, for all k. $\square$

Remark 2.10. We compute that $M_{24}^{s_4}$, $M_{24}^{s_5}$, $M_{24}^{s_7}$, $M_{24}^{s_9}$, $M_{24}^{s_{10}}$, $M_{24}^{s_{11}}$, $M_{24}^{s_{12}}$ and $M_{24}^{s_{13}}$ have 17, 18, 26, 20, 15, 15, 21 and 21 conjugacy classes, respectively. Hence, there are 502 possible pairs $(\mathcal{O}, \rho)$ for M_{24}; 492 of them have infinite-dimensional Nichols algebras, and 10 have negative braiding.

3. Using techniques based on non-abelian subracks

In the previous section, we discard the pairs $(\mathcal{O}, \rho)$ with $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ by mean of abelian subracks of $\mathcal{O}$. As a result, we showed that remain 23 pairs which give rise to a braided vector spaces with negative braiding. In this section, we consider these 23 remained pairs $(\mathcal{O}, \rho)$ and show that 16 of them lead to infinite-dimensional Nichols algebras using non-abelian subracks of $\mathcal{O}$ – see Subsection 1.3.

3.1. **The group M_{11}.** We have 5 remained cases.

CASE: $j = 10$. We choose in $\mathcal{O}_{s_{10}}$ the following elements: $\sigma_0 := s_{10}$, $\sigma_1 := (1, 6, 8)(2, 5, 3, 4, 10, 9)(7, 11)$ and $\sigma_2 := \sigma_0 \triangleright \sigma_1$. It is easy to see that the family $(\sigma_i)_{i \in \mathbb{Z}_3}$ is of type $\mathcal{D}_3$ in $\mathcal{O}_{s_{10}}$. Then $\dim \mathfrak{B}(\mathcal{O}_{s_{10}}, \chi_{(-1)}) = \infty$, by Lemma 1.9, with $p = 3$.

3.2. **The group M_{12}.** We have 4 remained cases.

CASE: $j = 14$. We choose in $\mathcal{O}_{s_{14}}$ the following elements: $\sigma_1 := s_{14}$, $\sigma_2 := (1, 3, 5, 11, 4, 8, 10, 12)(7, 9)$, $\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_{14}^3$ and $\tau_l := \sigma_l^5$, $1 \le l \le 6$. By straightforward computation we can check that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{O}^{(2)}$. Then, $\dim \mathfrak{B}(\mathcal{O}_{s_{14}}, \chi_{(-1)}) = \infty$ from Lemma 1.11.

CASE: $j = 5$. We choose in $\mathcal{O}_{s_5}$ the following elements: $\sigma_1 := s_5$, $\sigma_2 := (1, 4, 12, 7)(2, 6, 3, 8, 11, 10, 9, 5)$, $\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_5^3$ and $\tau_l := \sigma_l^5$, $1 \le l \le 6$. We can check that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{O}^{(2)}$. Then, $\dim \mathfrak{B}(\mathcal{O}_{s_5}, \chi_{(-1)}) = \infty$ from Lemma 1.11.

CASE: $j = 2$. We choose in $\mathcal{O}_{s_2}$ the following elements: $\sigma_0 := s_2$, $\sigma_1 := (1, 2, 12, 11, 8, 10)(3, 6, 9)(4, 5)$ and $\sigma_2 := \sigma_0 \triangleright \sigma_1$. It is easy to see that this family $(\sigma_i)_{i \in \mathbb{Z}_3}$ is of type $\mathcal{D}_3$ in $\mathcal{O}_{s_2}$. Then $\dim \mathfrak{B}(\mathcal{O}_{s_2}, \chi_{(-1)}) = \infty$ from Lemma 1.9, with $p = 3$.

3.3. **The group M_{22}.** We have one remained case. We choose in $\mathcal{O}_{s_4}$ the following elements: $\sigma_1 := s_4$,

$$\sigma_2 := (1, 5, 13, 10, 11, 7, 12, 22)(2, 9)(3, 21, 19, 15, 17, 18, 6, 8)(4, 14, 20, 16),$$

$\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_4^{-1}$ and $\tau_l := \sigma_l^5$, $1 \le l \le 6$. By straightforward computation we can check that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{O}^{(2)}$. Then, $\dim \mathfrak{B}(\mathcal{O}_{s_4}, \chi_{(-1)}) = \infty$ from Lemma 1.11.

In view of Theorem 2.5 and the previous paragraph we can state the following result.

Theorem 3.1. *Any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq M_{22}$ is necessarily isomorphic to the group algebra of M_{22}.* $\square$

3.4. **The group M_{23}.** We have 3 remained cases.

CASE: $j = 9$. We choose in $\mathcal{O}_{s_9}$ the following elements: $\sigma_1 := s_9$,

$$\sigma_2 := (1,3,5,20,10,14,13,23)(2,15,7,8)(4,22,12,6,17,16,21,11)(9,19),$$

$\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_9^{-1}$ and $\tau_l := \sigma_l^5$, $1 \leq l \leq 6$. We compute that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{D}^{(2)}$. Then, $\dim \mathfrak{B}(\mathcal{O}_{s_9}, \chi_{(-1)}) = \infty$ from Lemma 1.11.

3.5. **The group M_{24}.** We have 10 remained cases.

CASE: $j = 6$. The representative s_6 is

$$(1,9,20,17)(2,6)(3,10)(4,8)(5,24,19,7)(11,14,18,23)(12,21,13,15)(16,22).$$

We choose in $\mathcal{O}_{s_6}$ the following elements: $\sigma_1 := s_6$, σ_2 to be

$$(1,9,20,17)(2,11)(3,14)(4,18)(5,19,7,24)(6,10,8,22)(12,21,15,13)(16,23),$$

$\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := \sigma_2 \triangleright \sigma_3$, $\tau_1 := \sigma_6^{-1}$, $\tau_2 := \sigma_4^{-1}$, $\tau_3 := \sigma_5^{-1}$, $\tau_4 := \sigma_2^{-1}$, $\tau_5 := \sigma_3^{-1}$ and $\tau_6 := \sigma_1^{-1}$. By straightforward computation we can check that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{D}^{(2)}$. Now, assume that $\rho = \rho_{2,6}$ or $\rho_{3,6}$; then $q_{\sigma_1\sigma_1} = -1$ because $s_6 \in \mathcal{O}_{23}^{M_{24}^{s_6}}$. We define $g := (3,16)(5,12)(6,8)(7,15)(9,17)(13,19)(14,23)(21,24)$; we verify that $g \in M_{24}$ and that $g \triangleright \sigma_1 = \tau_1$. Also we compute that τ_1, $\sigma_6 \in \mathcal{O}_{24}^{M_{24}^{s_6}}$. Hence, from Table 16, we have that $\rho(\sigma_6) = \rho(\tau_1) = \rho(g\sigma_6 g) = -1$. Therefore, $\dim \mathfrak{B}(\mathcal{O}_{s_6}, \rho) = \infty$, from Lemma 1.10.

CASE: $j = 8$. The representative s_8 is

$$(1,4,24,14)(2,21,15,6)(3,16,8,12)(5,11,23,20)(7,18,17,13)(9,10,22,19).$$

We choose in $\mathcal{O}_{s_8}$ the following elements: $\sigma_0 := s_8$, σ_1 to be

$$(1,2,24,15)(3,5,8,23)(4,19,14,10)(6,22,21,9)(7,16,17,12)(11,13,20,18)$$

and $\sigma_2 := \sigma_0 \triangleright \sigma_1$. We compute that $(\sigma_i)_{i\in\mathbb{Z}_3}$ is a family of type $\mathcal{D}_3$ in $\mathcal{O}_{s_8}$. Now, if $\rho = \rho_{2,8}$ or $\rho_{2,8}$, then $q_{\sigma_0\sigma_0} = -1$, and $\dim \mathfrak{B}(\mathcal{O}_{s_{14}}, \rho) = \infty$ by from Lemma 1.9, with $p = 3$.

CASE: $j = 14$. The representative s_{14} is

$$(2,10,4,16,20,15,6,18)(3,17,11,5,7,12,21,13)(9,19)(14,24,23,22).$$

We choose in $\mathcal{O}_{s_{14}}$ the following elements: $\sigma_1 := s_{14}$,

$$\sigma_2 := (2,4,18,15,20,6,16,10)(3,12,13,21,7,17,5,11)(8,9)(14,22,24,23),$$

$\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_{14}^3$ and $\tau_l := \sigma_l^5$, $1 \leq l \leq 6$. By straightforward computation we can check that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{D}^{(2)}$. Now, if $\rho = \epsilon \otimes \chi_{(-1)}$ or $\mathrm{sgn} \otimes \chi_{(-1)}$, then $\dim \mathfrak{B}(\mathcal{O}_{s_{14}}, \rho) = \infty$, from Lemma 1.11.

CASE: $j = 17$. The representative s_{17} is

$$(1, 9, 12, 10, 17, 14, 3, 23, 5, 21, 19, 13)(2, 18)(4, 8, 15, 20)(6, 16, 7, 24, 22, 11).$$

We choose in $\mathcal{O}_{s_{17}}$ the following elements: $\sigma_1 := s_{17}$, σ_2 to be

$$(1, 9, 13, 3, 17, 14, 10, 19, 5, 21, 23, 12)(2, 18)(4, 8, 20, 15)(6, 11, 7, 16, 22, 24),$$

$\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_{17}^7$ and $\tau_l := \sigma_l^5$, $1 \leq l \leq 6$. We compute that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{O}^{(2)}$. Then, $\dim \mathfrak{B}(\mathcal{O}_{s_{17}}, \chi_{(-1)}) = \infty$ from Lemma 1.11.

CASE: $j = 18$. The representative s_{18} is

$$(1, 18, 20, 4, 17, 5, 24, 13, 11, 14, 7, 23)(2, 10, 16, 21, 22, 8, 15, 19, 12, 6, 9, 3).$$

We choose in $\mathcal{O}_{s_{18}}$ the following elements: $\sigma_1 := s_{18}$, σ_2 to be

$$(1, 2, 10, 14, 17, 22, 8, 18, 11, 12, 6, 5)(3, 15, 20, 4, 21, 9, 24, 13, 19, 16, 7, 23),$$

$\sigma_3 := \sigma_2 \triangleright \sigma_1$, $\sigma_4 := \sigma_3 \triangleright \sigma_1$, $\sigma_5 := \sigma_4 \triangleright \sigma_1$, $\sigma_6 := s_{18}^7$ and $\tau_l := \sigma_l^5$, $1 \leq l \leq 6$. We compute that the family $(\sigma_l)_{l=1}^6 \cup (\tau_l)_{l=1}^6$ is of type $\mathfrak{O}^{(2)}$. Then, $\dim \mathfrak{B}(\mathcal{O}_{s_{18}}, \chi_{(-1)}) = \infty$ from Lemma 1.11.

CASE: $j = 19$. The representative s_{19} is

$$(1, 5, 20, 23, 19, 2, 18, 7, 17, 9, 21, 24, 6, 12)(3, 14, 8, 15, 13, 11, 16)(4, 22).$$

We choose in $\mathcal{O}_{s_{19}}$ the following elements: $\sigma_0 := s_{19}$,

$$\sigma_1 := (1, 14, 20, 15, 19, 11, 18, 3, 17, 8, 21, 13, 6, 16)(2, 12, 7, 5, 9, 23, 24)(4, 10)$$

and $\sigma_2 := \sigma_0 \triangleright \sigma_1$. We compute that $(\sigma_i)_{i \in \mathbb{Z}_3}$ is a family of type $\mathcal{D}_3$ in $\mathcal{O}_{s_{19}}$. Then $\dim \mathfrak{B}(\mathcal{O}_{s_{19}}, \chi_{(-1)}) = \infty$, from Lemma 1.9, with $p = 3$ and $k = 9$..

CASE: $j = 20$. The representative is $s_{20} = s_{19}^{-1}$. Now, the family $(\sigma_i^{-1})_{i \in \mathbb{Z}_3}$, with σ_i as in the case $j = 19$ above, is of type $\mathcal{D}_3$. Then $\dim \mathfrak{B}(\mathcal{O}_{s_{20}}, \chi_{(-1)}) = \infty$, from Lemma 1.9, with $p = 3$ and $k = 9$.

In view of Theorem 2.9 and the previous paragraph we can state the following result.

Theorem 3.2. *Any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq M_{24}$ is necessarily isomorphic to the group algebra of M_{24}.* □

Remark 3.3. In the 7 remained cases, that appear in Table 1, we compute that there is no family of type $\mathfrak{O}^{(2)}$ nor $\mathcal{D}_p$, for any odd prime integer p, inside the respective conjugacy classes. Hence, in these cases we cannot decide whether the dimension of $\mathfrak{B}(\mathcal{O}, \rho)$ is infinite or not with the methods available today.

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